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Contents

Chapter 1

Introduction

1.1 Function theory

In this section I recollect some theorems from multi-variable function theory that are used in this paper. They are all quite elementary and omitted proofs can be found in [G].

Definition 1.1 *A subset $U \subseteq \mathbb{C}^n$ is called a* domain *if it is open and connected.*

Theorem 1.1 (Open mapping theorem) *Let $U \subseteq \mathbb{C}^n$ be a domain and*

$$F : U \to \mathbb{C}$$

a non-constant holomorphic function, then F is an open mapping.

Theorem 1.2 (Maximum principle) *Let $(\cdot, \cdot)$ denote the standard hermitian form on $\mathbb{C}^m$, and let $U \subseteq \mathbb{C}^n$ be a domain. If a holomorphic mapping*

$$F : U \to \mathbb{C}^m$$

is such that the (real valued) function

$$z \mapsto (F(z), F(z))$$

attains a maximum on U, then F is a constant mapping.

Proof: Suppose that $(F(z_o), F(z_o)) = M$ is maximal for some $z_o \in U$. Define a holomorphic function φ on U by:

$$\varphi(z) = (F(z), F(z_o))$$

Then the Schwarz inequality yields

$$|\varphi(z)|^2 \le (F(z), F(z)) \cdot M \le M^2$$

Now because $\varphi(z_o) = M$, the open mapping theorem implies that φ is constant on U. Again by Schwarz, we conclude that F maps into the circle $\Gamma_1 \cdot F(z_o)$, where Γ_1 is the unit circle in $\mathbb{C}$. By the open mapping theorem it follows that F has to be a constant.□

Definition 1.2 *Define*

$$\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$$

$$\Delta^* = \Delta \backslash \{0\}$$

and for $k \le m$:

$$\Delta^{m,k} = \{(z_1, \ldots, z_m) \in \Delta^m \mid z_j \ne 0 \textit{ for some } j \le k\}$$

Theorem 1.3 (Hartog's theorem) *Let m, k be two integers, $m \ge k \ge 2$, and*

$$F : \Delta^{m,k} \to \mathbb{C}$$

a holomorphic function. Then F extends to a holomorphic function on Δ^m.

Theorem 1.4 (Riemann extension theorem) *Let $m \ge 1$ be an integer and*

$$F : \Delta^{m,1} \to \mathbb{C}$$

a holomorphic function such that for any $w \in \Delta^{m-1}$ the function

$$z \mapsto F(z, w)$$

extends holomorphically to Δ. Then F extends holomorphically to Δ^m.

Theorem 1.5 (Isomorphism theorem) *Let m, k be two integers, $m \ge k \ge 2$. If a holomorphic mapping*

$$F : \Delta^{m,k} \to \mathbb{C}^m$$

is locally biholomorphic, then F extends to a locally biholomorphic mapping on Δ^m.

Proof: By Hartog's theorem, F extends holomorphically to Δ^m. The function

$$j = det\left(\frac{\partial F_i}{\partial z_j}\right)$$

is holomorphic on Δ^m and non-vanishing on $\Delta^{m,k}$. Hence $1/j$ is holomorphic on Δ^m. In particular, j is non-vanishing throughout Δ^m. This implies that F is locally biholomorphic. □

Definition 1.3 *If X is a topological space and $x \in X$ we denote its fundamental group with base point x by $\pi_1(X, x)$. If g_1 and g_2 are the homotopy classes of loops γ_1 and γ_2 respectively then $g_1 g_2$ is the homotopy class of the concatenation $\gamma_1 * \gamma_2$ obtained by passing along γ_1 and γ_2 in this order.*

Definition 1.4 *Let X and Y be connected complex manifolds. A surjective holomorphic map $\pi : X \to Y$ is called a* covering map *if every point $y \in Y$ has a neighborhood U such that the restriction of π to any connected component of $\pi^{-1}(U)$ is a biholomorphic map onto U.*

Let $\pi : X \to Y$ be a covering map. The set of all biholomorphic mappings g of X onto itself satisfying $\pi \circ g = g$ equiped with the product $(g_1, g_2) \mapsto g_1 \circ g_2$ is called the automorphism group *of the covering and is denoted by $Aut(X \mid Y)$.*

The cardinality of any fibre of π is called the degree *of the covering (this does not depend on the chosen fibre).*

A covering is called Galois *if its automorphism group acts transitively on each fibre.*

If X is simply connected then it is called a universal covering of Y.

Theorem 1.6 *Suppose $\pi : X \to Y$ is a universal covering map and $y \in Y$. The groups $\pi_1(Y, y)$ and $Aut(X \mid Y)$ are canonically isomorphic.*

If $\pi : X \to Y$ is a universal covering and $g \in \pi(Y, y)$ we write $x \mapsto gx$ for the corresponding covering automorphism.

Definition 1.5 *Let Y be an analytic variety (see [G]) and $D \subset Y$ be a subvariety such that $Y \backslash D$ is a complex manifold (i.e. is smooth). Let $\pi : X \to Y \backslash D$ be a Galois covering and take $y \in Y$. If U is a connected neighborhood of y such that $U \backslash D$ is connected then let $d(\pi, U)$ be the degree of the restriction of π to any connected component of $\pi^{-1}(U)$. The* local degree *of π at y is the minimum of $d(\pi, U)$ taken over all neighborhoods U as before.*

Definition 1.6 *Let X and Y be connected analytic manifolds. A surjective holomorphic map $\pi : X \to Y$ is called a* ramified covering *if it satisfies the following two conditions.*

1. *Every $y \in Y$ has a neighborhood U such that the restriction of π to any connected component of $\pi^{-1}(U)$ is a finite branched covering of U in the sense of [G].*

2. *If x_1 and x_2 are elements in X such that $\pi(x_1) = \pi(x_2)$ then there is a biholomorphic mapping g of X onto X such that $\pi \circ g = \pi$ and $g(x_1) = x_2$.*

The group of all biholomorphic mappings g of X onto X such that $\pi \circ g = \pi$ is called the automorphism group *of the covering and is denoted as $Aut(X \mid Y)$.*

Let $\pi : X \to Y$ be a ramified covering. The maximal cardinality of a fibre of π is called the degree *of the covering. If $y \in Y$ and U is a neighborhood of y then let $d(\pi, U)$ be the degree of π restricted to any connected component of $\pi^{-1}(U)$. The* local degree *of π at y is the minimum of all degrees $d(\pi, U)$ taken over all neighborhood s U of y.*

1.2 The symmetric group

Some notions and techniques used in this thesis will be introduced for the example of the root system of type A_n. This has the advantage that the associated reflection group is the symmetric group S_{n+1}. The structure of this group and its polynomial invariants will be familiar to the reader. Nevertheless, even for this case one can prove non-trivial results. Studying the symmetric group leads to an intrinsic proof of a theorem by Orlik and Solomon [OS] on the invariants of Shephard groups related to S_{n+1} and a result of Coxeter [C] on presentations of such groups. The proofs in [OS] and [C] are based on a case by case verification using a computer.

Consider the symmetric group S_{n+1} for some $n \geq 1$. It has a natural representation ρ on $\mathbb{C}^{n+1}$. If $e_1, \ldots, e_{n+1}$ is the canonical basis of $\mathbb{C}^{n+1}$ and $\sigma \in S_{n+1}$ then $\rho(\sigma)e_j = e_{\sigma(j)}$.

Let $z_1, \ldots, z_{n+1}$ denote the canonical linear coordinates on $\mathbb{C}^{n+1}$. It is well known that the algebra $P[\mathbb{C}^{n+1}]^{S_{n+1}}$ of symmetric polynomials on $\mathbb{C}^{n+1}$ is generated by the elementary symmetric polynomials $s_1, \ldots, s_{n+1}$. These are defined by

$$\prod_{j=1}^{n+1}(X - z_j) = X^{n+1} - s_1 X^n + \ldots + (-1)^{n+1} s_{n+1}$$

in particular $s_1 = z_1 + \ldots + z_{n+1}$ and s_j is homogeneous of degree j. The restriction of ρ to the n-dimensional subspace V given by $s_1 = 0$ is irreducible. In the rest of this introduction we use this restriction. The square of the polynomial δ on V given by

$$\delta = \prod_{1 \leq i < j \leq n+1} (z_i - z_j)$$

is clearly symmetric. Hence $\delta^2 = D(s_2, \ldots, s_{n+1})$ for some polynomial $D \in \mathbb{C}[x_1, \ldots, x_n]$ in the indeterminates $x_1, \ldots, x_n$. This D is called the *discriminant* of S_{n+1}. Note that δ vanishes at $z \in V$ if and only if $z_i = z_j$ at z for some $i \neq j$, i.e. if and only if z is fixed by $\rho(i\ j)$. The complement of the vanishing locus of δ is called V^{reg}, a point in this complement is called *regular*. Note that a point is regular precisely if its S_{n+1}-orbit contains $(n+1)!$ points. The vanishing locus of D on $\mathbb{C}^n$ is denoted by Δ or Δ^{n-1} to indicate the dimension.

The set Δ has a natural stratification as follows. Let $(a_1, \ldots, a_m)$ be a non-decreasing sequence of integers such that $a_1 > 1$ and $|a| := a_1 + \ldots + a_m \leq n+1$. Let $\Delta_{(a_1,\ldots,a_m)}$ denote the stratum of all points $(x_1, \ldots, x_n) \in \Delta$ such that the polynomial

$$X^{n+1} + x_1 X^{n-1} - \ldots + (-1)^{n+1} x_n$$

has exactly m multiple zeroes with multiplicities $a_1, \ldots, a_m$ respectively. For example $\Delta_{(n+1)} = \{0\}$ and $\Delta_{(2)}$ is the "subregular" stratum of dimension $n-1$. If $x \in \Delta_{(a_1,\ldots,a_m)}$ then there exists a coordinate neighborhood of x that is isomorphic to a Cartesian product of $m+1$ factors of the following kind: An $(n+m-|a|)$-dimensional polydisc and for each $1 \leq j \leq m$ the complement of Δ^{a_j-2} in an (a_j-1)-dimensional polydisc. We will make use of this local structure later on in an inductive argument on the dimension n.

Using the elementary symmetric polynomials as coordinates we get a map

$$S : V \to \mathbb{C}^n, \; S : z \mapsto (s_2(z), \ldots, s_{n+1}(z)).$$

To study the complement $\mathbb{C}^n \backslash \Delta$ fix a base point $u = (u_1, \ldots, u_{n+1}) \in V$ such that $u_j \in \mathbb{R}$ for all j and $u_1 < u_2 < \ldots < u_{n+1}$. For $j = 1, \ldots, n$ define a path γ_j connecting u with $\rho(j\ j{+}1)u$ as follows

$$\gamma_j(t) = u + \frac{1 - e^{\pi i t}}{2}(\rho(j\ j{+}1)u - u), \; t \in [0,1].$$

Theorem 1.7 *The fundamental group $G := \pi_1(\mathbb{C}^n \backslash \Delta, u)$ is generated by the homotopy classes g_j of the loops $S \circ \gamma_j$. Moreover it has the following presentation*

$$\begin{array}{lcl} \langle g_1, \ldots, g_n & | & g_i g_j = g_j g_i, \text{ if } 1 \leq i,j \leq n \text{ and } |i-j| > 1 \\ & & g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}, \text{ all } 1 \leq j < n \rangle \end{array}$$

Proof: See [FN]. □

The group G is isomorphic to the braid group of $n+1$ strings as introduced by Artin [A1,A2]. For any integer $p \geq 2$ we denote the smallest normal subgroup of G containing all elements g_j^p by $\Gamma(p)$. The quotient $G/\Gamma(p)$ is called a *truncated braid group*. We can now prove an important geometric property of the map S.

Theorem 1.8 *The map S is a branched covering map with branch locus Δ. The restriction of S to V^{reg} is a Galois covering of $\mathbb{C}^n \backslash \Delta$ of local degree two along $\Delta_{(2)}$. Moreover it is universal with respect to this property.*

Proof: That S is a covering map with branch locus Δ follows from the fact that we can recover z from $(s_2(z), \ldots, s_{n+1}(z))$ upto the S_{n+1}-action. Moreover S_{n+1}

acts transitively on the fibres. This also shows that the local degree along $\Delta_{(2)}$ is two. The universal covering of $\mathbb{C}^n\backslash\Delta$ has an automorphism group isomorphic to G. Now it is well known that $S_{n+1} \cong G/\Gamma(2)$. This shows that the covering S is universal. $\Box$

This nice theorem gives rise to the following question. For which $p \geq 3$ is the universal Galois covering of $\mathbb{C}^n\backslash\Delta$ of local degree p along $\Delta_{(2)}$ a *finite* covering and what is the structure of such a covering?

In this introduction we will sketch a proof of the following result.

Theorem 1.9 *Suppose $p \geq 3$ is such that $1-(n+1)(1/2-1/p) > 0$. Then the truncated braid group $G/\Gamma(p)$ has a faithful representation ρ_p on an n-dimensional complex vector space E such that the image $G(p) \leq End(E)$ is finite and generated by complex reflections $\rho_p(g_j)$ of order p. Moreover there are homogeneous $h_1,\ldots,h_n \in P[E]$ generating $P[E]^{G(p)}$ such that $(h_1,\ldots,h_n) : E \to \mathbb{C}^n$ is a ramified covering with branch locus Δ and of local degree p. All possibilities are listed in the following table:*

n	1	2	3	4
p	≥ 3	$3,4,5$	3	3

Proof: The proof is a combination of linear algebra and complex analysis. The idea is to construct a function of Nilsson class [D] of determination order n on V^{reg} with some S_{n+1}-invariance and homogeneity properties. This induces a multivalued map $ev : \mathbb{C}^n\backslash\Delta \to E$ for some complex vector space E and the representation ρ_p by analytic continuation. Then it is proved that ev has a *single valued* inverse h on E which is polynomial and ρ_p-invariant, proving the theorem.

The case $n=1$ is trivial. Ramified coverings of any positive local degree at $0 \in \mathbb{C}$ are given by the maps $x \mapsto x^p$. Therefore we assume that n is at least 2.

Let $U \subset V^{reg}$ be a simply connected neighborhood of u that does not intersect any other of its S_{n+1}-conjugates. Take $k \in [0,1/2)$ and $z=(z_1,\ldots,z_{n+1}) \in U$ with real coordinates. Define a holomorphic differential form $\phi(k;z)$ on the extended upper half plane

$$\mathcal{H}_z := \{s \in \mathbb{C} \mid \mathrm{Im}(s) \geq 0,\ s \neq z_j,\ j=1,\ldots,n+1\}$$

by

$$\phi(k;z) := \prod_{j=1}^{n+1} (z_j - s)^{-k} ds$$

where we take $a^b := \exp(b \cdot \log a)$ for $a > 0$. For each $j=1,\ldots,n$ define a function $f_j(k;\cdot)$ on U by

$$z \mapsto f_j(k;z) := \int_{z_j}^{z_{j+1}} \phi(k;z).$$

Note that in case $k = 0$ these are just n independent linear functions on V.

Lemma 1.1 *The following properties for the $f_j(k;\cdot)$ hold:*

1. *Any $f_j(k;\cdot)$ extends to a multivalued holomorphic function on V^{reg}.*
2. *The functions $f_1(k;\cdot), \ldots, f_n(k;\cdot)$ on U are linearly independent over $\mathbb{C}$.*
3. *The $\mathbb{C}$-vector space $F(k)$ spanned by the $f_j(k;\cdot)$ is invariant under analytic continuation.*
4. *Each $f_j(k;\cdot)$ is homogeneous of degree $1-(n+1)k$.*
5. *If $\sigma \in S_{n+1}$ and $f[\gamma]$ is the analytic continuation of $f \in F(k)$ to $\rho(\sigma)U$ along a path γ connecting u and $\rho(\sigma)u$ then $z \mapsto f[\gamma](\rho(\sigma)z)$ is again an element of $F(k)$.*

Proof: All statements except 2 can be easily verified using the definition of the $f_j(k;\cdot)$. A proof of 2 is given in the next chapter. □

From these properties we conclude that the map S induces an n-dimensional vector space $F_S(k)$ of functions on $S(U)$, spanned by the functions

$$e_j(k;\cdot) := e^{-\pi ijk} f_j(k;\cdot)\circ(S|_U)^{-1}.$$

This space is invariant under analytic continuation along loops in $\mathbb{C}^n\backslash\Delta$. The resulting *right* representation

$$M_k : G \to \operatorname{End}(F_S(k))$$

is called the *monodromy representation*.

Define q as $\exp(-2\pi ik)$ and $q^{1/2} = \exp(-\pi ik)$. We omit the proofs of the following two theorems.

Theorem 1.10 *For $1 \le i,j \le n$, $i \neq j$ we have:*

1. *$M_k(g_j)e_j(k;\cdot) = -qe_j(k;\cdot)$.*
2. *$M_k(g_j)e_i(k;\cdot) = e_i(k;\cdot)$ if $|i-j| > 1$.*
3. *$M_k(g_j)e_i(k;\cdot) = e_i(k;\cdot) + q^{1/2}e_j(k;\cdot)$ if $|i-j| = 1$.*

In particular $M_k(g_j)$ is a complex reflection *on $F_S(k)$.*

Theorem 1.11 *With respect to the basis $e_j(k;\cdot)$, $j = 1, \ldots, n$, the following $n \times n$ matrix defines an M_k-invariant Hermitian structure on $F_S(k)$.*

$$H_k := \begin{pmatrix} 2\cos(\pi k) & -1 & & & \emptyset \\ -1 & 2\cos(\pi k) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2\cos(\pi k) & -1 \\ \emptyset & & & -1 & 2\cos(\pi k) \end{pmatrix}$$

The Hermitian form H_k is positive definite iff $1 - (n+1)k > 0$.

Note that $M_k(g_j)$ and H_k can be interpreted as a deformation in k of the generating reflections of ρ and the Hermitian form on V.

Denote the dual of $F_S(k)$ by E_k. There is a canonical map

$$ev : S(U) \to E_k,\ x \mapsto \text{evaluation at } x$$

called the *evaluation map*. It can be continued analytically along any path in $\mathbb{C}^n \backslash \Delta$ so we will think of it as a multivalued function on this space. This map is *weighted homogeneous* on $\mathbb{C}^n \backslash \Delta$, i.e.

$$\mathrm{ev}(\lambda^2 x_1, \lambda^3 x_2, \ldots, \lambda^{n+1} x_n) = \lambda^{1-(n+1)k} \mathrm{ev}(x_1, \ldots, x_n)$$

for any $\lambda \in \mathbb{C}^*$. Its local properties are as follows.

Theorem 1.12 *The evaluation map is everywhere locally biholomorphic. For any continuation of* ev *near a point $p \in \Delta_{(2)}$ there are local coordinates $y_1, \ldots, y_n$ near p and linear coordinates on E_k such that Δ has local equation $y_1 = 0$ and the evaluation map is given by*

$$x \mapsto (y_1^{1/2-k}, y_2, \ldots, y_n)$$

for x near p.

Proof: An argument involving the explicit integral formulas for $e_j(k;\cdot)$ and the so called *Wronskian* of the function space $F_S(k)$. Details can be found in the next two chapters. □

Let M_k^* denote the transpose of M_k on E_k, i.e.

$$(M_k^*(g)\lambda)(f) = \lambda(M_k(g)f)$$

for all $\lambda \in E_k$ and $f \in F_S(k)$. Then M_k^* is a *left* representation.

If $\pi : \widetilde{X} \to \mathbb{C}^n \backslash \Delta$ is the universal covering then ev extends to a single valued holomorphic map $\widetilde{ev}$ on $\widetilde{X}$ and satisfies

$$\widetilde{\mathrm{ev}}(g \cdot x) = M_k^*(g)\widetilde{\mathrm{ev}}(x)$$

for all covering automorphisms $g \in G$. We denote the image of G under M_k^* by G_k.

Let $p \geq 3$ and $k = 1/2 - 1/p$ be such that $H_k > 0$, i.e. $1-(n+1)k > 0$. Then $\widetilde{ev}$ is $\Gamma(p)$ invariant and descends to a single valued function ev_u on the universal Galois covering

$$\pi_u : X_u(p) := \Gamma(p)\backslash\widetilde{X} \to \mathbb{C}^n \backslash \Delta$$

of local degree p along $\Delta_{(2)}$. In particular $\Gamma(p)$ is contained in the kernel of M_k^*.

Considering the local structure of $\mathbb{C}^n \backslash \Delta$ near some point $\neq 0$ in Δ one can prove by induction on the rank n that $X_u(p)$ embeds in a ramified covering $\pi_r : X_r(p) \to \mathbb{C}^n \backslash \{0\}$ with branch locus $\Delta \backslash \{0\}$. This means that $X_u(p) = \pi_r^{-1}(\mathbb{C}^n \backslash \Delta)$ and $\pi_u = \pi_r|_{X_u(p)}$. Moreover ev_u extends to a *locally biholomorphic* map ev_r on $X_r(p)$.

The Hermitian form H_k on $F_S(k)$ induces an M_k^*-invariant metric on E_k. By an elementary topological argument and homogeneity of ev_r one deduces the following. There exists a positive number $\epsilon > 0$ such that any local inverse of ev_r near a point $y \in E_k$ extends to a ball centered at y with radius ϵ times the distance of y to 0. Hence any local inverse extends to $E_k \backslash \{0\}$ because this is a simply connected set if $n \geq 2$. This shows that ev_r is an isomorphism between $(X_r(p), G/\Gamma(p))$ and $(E_k \backslash \{0\}, G_k)$.

Now the map h on $E_k \backslash \{0\} \to \mathbb{C}^n \backslash \{0\}$ given by $h := \pi_r \circ ev_r^{-1}$ is a map having holomorphic functions on $E_k \backslash \{0\}$ as coordinates. Because 0 is of co-dimension at least two in E_k, Hartog's theorem implies that h extends holomorphically to E_k and clearly $h(0) = 0$. In particular $h : E_k \to \mathbb{C}^n$ is a ramified covering of *finite degree* and its automorphism group G_k is finite.

Note that for any j the j^{th} coordinate h_j of h is homogeneous of degree

$$\frac{j+1}{1-(n+1)k}$$

which must therefore be an integer. This implies that $1-(n+1)k$ itself equals $1/m$ for some integer $m \geq 2$ and each h_j is a *polynomial*. Moreover each h_j is G_k invariant and in fact they generate the algebra of G_k-invariant polynomials on E_k.

Taking $\rho_p := M_k^*$, $E := E_k$ and $G(p) := G_k$, the theorem is proved. □

The isomorphism $G(p) \cong G/\Gamma(p)$ gives a presentation of $G(p)$ consisting of the braid relations for G and a relation for each generator to make its order p. The existence of generators $h_1, \ldots, h_n$ for $P[E]^{G(p)}$ such that h is a branched covering with branch locus Δ is a special case of a result in [OS] on discriminants of Shephard groups.

1.3 Literature

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Chapter 2

Lauricella's F_D

2.1 Abstract

In this chapter we study the moduli space of (multivalued) differential forms on $\mathbb{P}^1$ with $n+3$ singular points with fixed exponents, for some $n \geq 1$. Integration of such a form gives rise to a *period* or *evaluation* mapping closely related to the hypergeometric function F_D of Lauricella in n variables. If one imposes some conditions on the exponents at the singularities of the form this evaluation mapping establishes an isomorphism between a certain geometric quotient $(\mathbb{P}^1)^{n+3}/\mathrm{PGL}(2, \mathbb{C})$ and a quotient B/Γ of a complex hyperbolic ball where Γ is induced by monodromy of the evaluation mapping.

2.2 Introduction

The classical hypergeometric function was already studied by Euler in the 18^{th} century. More famous are the impressive results Gauss obtained concerning this function, which is also referred to as *Gauss' hypergeometric function*. The subject of this chapter was initiated in the 19^{th} century by Riemann [R] and Schwarz [S]. Riemann found a particularly nice way to study properties like monodromy and transformation formulae for the Gauss function. Schwarz then found all parameters for which the Gauss function has a finite monodromy group, i.e. for which it is *algebraic*. His methods were geometric of nature and later Klein generalized his work to obtain *discrete* monodromy groups (related to the so-called Klein triangle groups).

After this, generalizations have been carried out in two directions. In 1989 (!) Beukers and Heckman [BH] found the parameters for which the higher hypergeometric function ${}_nF_{n-1}$ has finite monodromy. In this direction, the question remains when this higher hypergeometric function has *discrete* monodromy. A question which is, as far as I know, not yet answered. The second direction of generalizations of the classical work was in *several variables*. Hypergeometric functions of two variables were introduced by Appell [A,AK]. Picard then used Appell's function F_1 (Appell introduced F_1 up to F_4) to study the same questions about finiteness and discreteness of its monodromy. Though he couldn't settle these questions in detail (in fact some of his arguments were wrong) he did some important work on this function [P1..3].

Little after Appell, Lauricella [L] gave a generalization of the functions $F_1 ... F_4$ in arbitrarily many variables called F_D, F_A, F_B, F_C respectively. In the $1970's$ Terada [T] used the Lauricella F_D to continue Picard's work. But he also did not have the complete proofs, though he did get the right answers. Then some ten years later the famous paper by Deligne and Mostow was published [DM]. They investigated the same questions as Terada and have given a rigorous treatment of the subject. Recently a very nice paper by Thurston [Th] was published in which he studies a related moduli problem but now using combinatorial techniques and theory of conic manifolds. The word *hypergeometric function* does not appear in his paper.

The intention of this chapter is to combine some ideas found in [DM] and [Th] to get a fairly elementary treatment of the subject. I would like to thank G. Heckman, for many fruitful discussions, E. Looijenga for introducing me to the subject of Geometric Invariant Theory and H. de Vries for careful reading of the manuscript.

2.3 The hypergeometric function F_D

Let $n \in \mathbb{N}$ be at least 1. In this section we fix parameters $\mu_0, \ldots, \mu_{n+2} \in (0, 1)$ such that $\sum \mu_j = 2$. Take real numbers $z_1^o, \ldots, z_n^o$ such that $0 < z_1^o < \ldots < z_n^o < 1$ and let z^o be the point $(z_1^o, \ldots, z_n^o)$ on $(\mathbb{P}^1)^n$ where we think of $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$. This point will serve later on as a base point for some fundamental group etc. Now take $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and assume for the moment that also $0 < z_1 < \ldots < z_n < 1$. We sometimes denote $0, 1, \infty$ as z_0, z_{n+1}, z_{n+2} respectively. Subscripts should be taken mod $n+3$ hence $\mu_{n+3} = \mu_0$ etc.

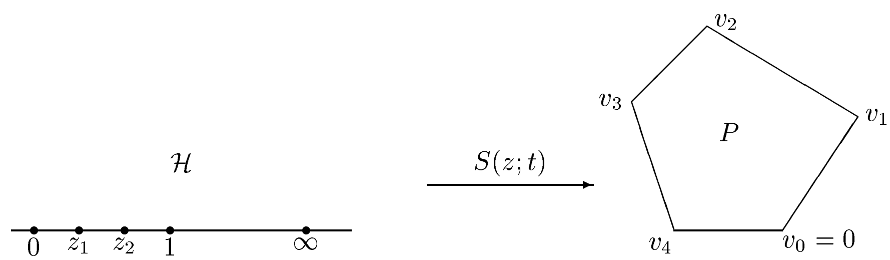
Define on the union of the upper half plane $\mathcal{H}$ with the intervals (z_j, z_{j+1}), $j = 0, \ldots, n+2$ the holomorphic function

$$\varphi(z_1, \ldots, z_n) : s \mapsto \varphi(z_1, \ldots, z_n; s) = \prod_{j=0}^{n+1} (z_j - s)^{-\mu_j}$$

such that $\varphi(z; s) > 0$ if $s < 0$. The exponent of the differential $\varphi(z; s)ds$ at infinity equals $-\mu_{n+2}$. Integrating this form along curves in $\mathcal{H}$ yields the so called Schwarz-Christoffel mapping on $\mathcal{H}$:

$$S(z;t) = \int_0^t \varphi(z;s)ds$$

Here we integrate along any path through $\mathcal{H}$ connecting 0 and t. This mapping can be described geometrically in a very nice way. It maps $\mathcal{H}$ biholomorphically onto the interior of a polygon $P(z)$ with vertices (in counter clockwise order) $v_j = S(z;z_j)$. At vertex v_j the interior angle equals $(1-\mu_j)\pi$, so by our choice of the parameters μ_j, the polygon $P(z)$ will be *convex*.



For $j \in \{0,\ldots,n+2\}$ let $e_j(z) = v_{j+1} - v_j$ be the j-th directed edge of $P(z)$, or:

$$e_j(z) = \int_{z_j}^{z_{j+1}} \varphi(z;s)ds$$

This integral formula shows that edges are analytic functions of their argument z near the basepoint z^o and can be continued analytically throughout $X \subset (\mathbb{P}^1)^n$ given by:

$$X = \{(u_1,\ldots,u_n) \in (\mathbb{P}^1)^n \mid \#\{0,u_1,\ldots,u_n,1,\infty\} = n+3\}$$

Clearly the sum of all $n+3$ edges equals zero, but there is even a stronger dependence.

Lemma 2.1 *As analytic functions of the parameter z, any set of $(n+2)$ edges is linearly dependent (over $\mathbb{C}$).*

Proof: Let E be a set of $(n+2)$ edges and let e_j be the edge that is not contained in E. Take z near z^o and real valued. Reflect $P(z)$ in the edge connecting v_j and v_{j+1} and glue the image to $P(z)$. The directed edges of this bigger polygon are exactly e_k and $c_k e_k$ for $k \neq j$ and some $n+2$ constants (i.e. not depending on z) on the unit circle, hence

$$\sum_{k\neq j} (1+c_k)e_k(z) = 0$$

is a non trivial linear relation on which holds independently of z. $\square$

Not all $(n+1)$-tuples of edges need to be linearly independent. However, we will show that an independent set of $(n+1)$ edges always exists.

Theorem 2.1 *Let $J = \{0, \ldots, n+2\} \backslash \{k_1, k_2\}$ be a set of $n+1$ elements. The edges e_j, $j \in J$ are linearly dependent iff both edges e_{k_1} and e_{k_2} are "parallel", i.e. iff*

$$\mu_{k_1+1} + \mu_{k_1+2} + \ldots + \mu_{k_2} = 1.$$

(This means that if z is chosen real valued then the edges $e_{k_{1,2}}(z)$ of $P(z)$ are really parallel.)

Proof: The "if" part follows from the proof of the previous lemma (if remaining edges are parallel, some c_k will equal -1). Now suppose that the remaining edges are *not* parallel. Then one checks that any small variation of the lengths of the edges of $P(z^o)$ with indices in J still realizes a convex polygon P' (without changing the interior angles). Of course the lengths of the remaining two edges are then completely determined. Because any polygonal domain is the biholomorphic image of $\mathcal{H}$ under a Schwarz-Christoffel mapping, it follows that there are numbers $0 < w_1 < \ldots < w_n < 1$ such that the mapping $S(w;t)$ maps $\mathcal{H}$ onto the interior of a convex polygon which is affinely isomorphic to P'. These $n+1$ degrees of freedom show that the edge functions with index in J are linearly independent. □

The edge $e_{n+1}(z)$ (upto a scalar) is known as the *Lauricella hypergeometric function F_D*. It is a generalization of the Gauss function in several variables. Taylor expansion at 0 using Euler's B function yields:

$$e^{\pi i \mu_{n+2}} e_{n+1}(z) = e^{\pi i \mu_{n+2}} \int_1^\infty \varphi(z;s) ds =$$

$$= \frac{\Gamma(1-\mu_{n+2})\Gamma(1-\mu_{n+1})}{\Gamma(\mu_0 + \ldots + \mu_n)} \sum_{m \in \mathbb{N}^n} \frac{(1-\mu_{n+2})_{|m|}(\mu)_m}{(\mu_0 + \ldots + \mu_n)_{|m|} m!} z^m$$

Here multi index notation is used and moreover:

$$(\mu)_m = (\mu_1)_{m_1} (\mu_2)_{m_2} \cdots (\mu_n)_{m_n}$$

$$m! = m_1! m_2! \cdots m_n!$$

The above sum is absolutely convergent if $|z_j| < 1$ for all j and is denoted by

$$F_D(1-\mu_{n+2}, \mu_1, \ldots, \mu_n, \mu_0 + \ldots + \mu_n; z_1, \ldots, z_n)$$

Note that if $n = 1$, we have:

$$F_D(\alpha, \beta, \gamma; z_1) = F(\alpha, \beta, \gamma; z_1)$$

Remark 2.1 *Define $n+1$ differential operators as follows:*

$$\theta_j := z_j \frac{\partial}{\partial z_j},\ j = 1, \ldots, n,\ \Theta := \theta_1 + \ldots + \theta_n$$

Let $F_D := F_D(\alpha, \beta_1, \ldots, \beta_n, \gamma; z_1, \ldots, z_n)$. Then one deduces from its power series expansion

$$[(\Theta + \gamma - 1)\theta_j - z_j(\Theta + \alpha)(\theta_j + \beta_j)]\, F_D = 0$$

for all $j = 1, \ldots, n$. The local solution space of these n equations at any non singular point is $(n+1)$-dimensional and spanned by the edges $e_j(z)$ for suitable parameters μ. A solution f near z is completely determined by prescribing the values of f and all its (first order) partial derivatives at z.

For any $z \in X$ let $M(z)$ denote the punctured Riemann sphere, $M(z) = \mathbb{P}^1 \backslash \{0, 1, z_1, \ldots, z_n, \infty\}$. On $M(z)$ the volume form $\Omega(z) = (i/2)|\varphi(z;s)|^2 ds \wedge \overline{ds}$ is well defined (because all μ_j are *real*). This form can be considered as the pull back of the euclidean volume on $\mathbb{C}$ by a "Schwarz-Christoffel" mapping (which is clear if z is real valued). The volume

$$\mathrm{Vol}(M(z)) = \int_{M(z)} \Omega(z)$$

is positive and finite. It can be expressed in a nice way using the vertices $v_j(z)$. To do so, we introduce and study the notion of *Area* of loops in $\mathbb{C}$.

Definition 2.1 *Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise smooth loop (so $\gamma(0) = \gamma(1)$). We define the* Area *of this loop by:*

$$Area(\gamma) = \frac{1}{2i} \int_\gamma \overline{z} dz$$

The area of a loop is just the euclidean area of the region in $\mathbb{C}$ that is enclosed in this loop. (Every point is counted as many times as the loop winds around it in counter clockwise direction). The area of a loop gives rise to a hermitean form on a certain $(n+1)$-dimensional space. Define $\beta_j = \exp(\pi i \mu_j)$ and $\omega_j = \beta_0 \cdots \beta_j$ for $j \in \{0, \ldots, n+2\}$. Then $|\omega_j| = 1$ and $\omega_{n+2} = 1$. If $w_0, \ldots, w_{n+2}$ are the canonical linear coordinates on $\mathbb{C}^{n+3}$ let $\mathrm{Pol}(\mu) \subset \mathbb{C}^{n+3}$ be the $(n+1)$-dimensional $\mathbb{C}$-linear subspace defined by the linearly independent equations

$$\sum_{j=0}^{n+2} \omega_j w_j = \sum_{j=0}^{n+2} \overline{\omega}_j w_j = 0$$

The $\mathbb{R}$-linear subspace

$$\mathrm{Pol}_{\mathbb{R}}(\mu) := \mathrm{Pol}(\mu) \cap \mathbb{R}^{n+3}$$

is a real form of $\mathrm{Pol}(\mu)$. To a vector $w \in \mathrm{Pol}(\mu)$ we associate two piecewise linear loops $P_+(w)$ and $P_-(w)$ which pass through the points

$$(0, \omega_0 w_0, \omega_0 w_0 + \omega_1 w_1, \ldots, \omega_0 w_0 + \ldots + \omega_{n+1} w_{n+1})$$

and

$$(0, \overline{\omega}_0 w_0, \overline{\omega}_0 w_0 + \overline{\omega}_1 w_1, \ldots, \overline{\omega}_0 w_0 + \ldots + \overline{\omega}_{n+1} w_{n+1})$$

respectively in the given order. Note that if $w \in \mathrm{Pol}_{\mathbb{R}}(\mu)$ then $P_-(w)$ is the complex conjugate of $P_+(w)$ and vice versa. We can now define an hermitian structure H on $\mathrm{Pol}(\mu)$ by

$$H(w, w) = \mathrm{Area}(P_+(w)) - \mathrm{Area}(P_-(w))$$

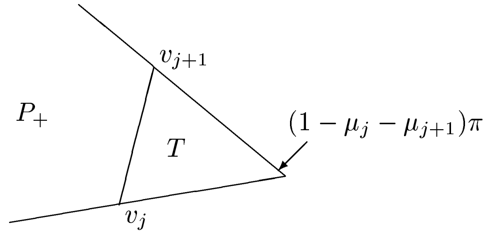
for all $w \in \mathrm{Pol}(\mu)$. In particular, if $w \in \mathrm{Pol}_{\mathbb{R}}(\mu)$ then $H(w, w) = 2\mathrm{Area}(P_+(w))$. If v and w are both in $\mathrm{Pol}(\mu)$ then by triangulating these polygons one can compute explicitly (coordinates of w and v indexed from 0 to $n+2$):

$$H(v, w) = \frac{1}{4i} \sum_{0 \le k < l \le n+1} (\overline{\omega}_k \omega_l - \omega_k \overline{\omega}_l)(v_k \overline{w}_l + v_l \overline{w}_k)$$

Note that H restricted to $\mathrm{Pol}_{\mathbb{R}}(\mu)^2$ is *real valued*. We will exploit this fact in the proof of the following theorem.

Theorem 2.2 *Let n be at least 2. Then the hermitean form H on Pol(μ) is* hyperbolic, *i.e. has signature $(1, n)$.*

Proof: By the remark above, it suffices to show that the restriction of H on $\mathrm{Pol}_{\mathbb{R}}(\mu)^2$ is hyperbolic. We will diagonalize this restriction step by step. Take $w \in \mathrm{Pol}(\mu) \cap \mathbb{R}^{n+3}_{>0}$. Because $n \ge 2$ there is a number $j \in \{0, \ldots, n+2\}$ such that $\mu_j + \mu_{j+1} < 1$. This implies that there are positive real numbers x, y such that $\omega_j = x\omega_{j-1} + y\omega_{j+1}$. Note that x and y do *not* depend on w but only on μ. Now the part of the polygon $P_+(w)$ near the j^{th} edge looks like an angle with a triangle T clipped off:



The lengths of the edges of T are w_j, xw_j and yw_j respectively. The angle of T opposite to the j^{th} edge of $P_+(w)$ measures $(1-\mu_j-\mu_{j+1})\pi$. Together this shows

$$\mathrm{Area}(T) = \tfrac{1}{2}xy\sin((\mu_j+\mu_{j+1})\pi)w_j^2 =: \tfrac{1}{2}tw_j^2$$

Let the $n+2$ parameters $\mu'_0, \ldots, \mu'_{n+1}$ be given by

$$\mu_0, \ldots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \ldots, \mu_{n+2}$$

respectively and $w' \in \mathbb{R}^{n+2}$ by

$$w' = (w_0, \ldots, w_{j-1} + xw_j, w_{j+1} + yw_j, \ldots, w_{n+2})$$

Then $w' \in \mathrm{Pol}_{\mathbb{R}}(\mu')$ and if we glue T to $P_+(w)$ we obtain the bigger polygon $P_+(w')$ having one vertex and edge less. Now $t > 0$ and clearly

$$2\mathrm{Area}(P_+(w)) = 2\mathrm{Area}(P_+(w')) - (\sqrt{t}w_j)^2$$

Repeat this procedure of sticking on triangles until, after n such steps, we reach a polygon $P_+(w'')$ which is itself a triangle. (One only has to take care to avoid a parallellogram on the way). The lengths of the edges of this triangle are all positive linear combinations of $w_0, \ldots, w_{n+2}$. Its area is quadratic in any length of an edge. So we constructed $n+1$ real functionals $f_0, \ldots, f_n$ on $\mathrm{Pol}_{\mathbb{R}}(\mu)$ such that

$$H(w,w) = 2\mathrm{Area}(P_+(w)) = f_0(w)^2 - f_1(w)^2 - \ldots - f_n(w)^2$$

and by considering each reduction step we conclude that these functionals are linearly independent.

Because $\mathrm{Pol}(\mu) \cap \mathbb{R}^{n+3}_{>0}$ is open in $\mathrm{Pol}_{\mathbb{R}}(\mu)$ we conclude that the latter equality holds throughout $\mathrm{Pol}_{\mathbb{R}}(\mu)$ if this open cone would be non empty. Now 0 is contained in the convex hull of the ω_j and any realisation of 0 as a convex combination with all positive coefficients yields a non zero element of the open cone above. Hence the restriction of H to this real form is hyperbolic and hence H is itself hyperbolic.□

Here is how the volume of $M(z)$ relates to this hermitean form.

Theorem 2.3 *For all $z \in X$ the following equality holds:*

$$\mathrm{Vol}(M(z)) = H(w(z), w(z)),$$

where $w(z) \in \mathrm{Pol}(\mu)$ is given by

$$w(z) = (\overline{\omega}_0 e_0(z), \overline{\omega}_1 e_1(z), \ldots, \overline{\omega}_{n+2} e_{n+2}(z))$$

and the edge functions $e_0, \ldots, e_{n+2}$ are continued analytically along any path from $z^o \in X$ to $z \in X$.

Proof: Take z in X. Let $\Gamma : [0,1] \to X$ be any smooth path in X connecting z^o and z. We deform the half line $[0,\infty]$ accordingly: Let $\gamma : [0,1] \times [0,1] \to \mathbb{P}^1(\mathbb{C})$ be continuous such that

1. $\gamma(s,\cdot)$ is a smooth non self-intersecting curve for all $s \in [0,1]$.
2. $\gamma(s,0) = 0$, $\gamma(s,1) = \infty$ and $\gamma(s,\cdot)$ passes through the points $\Gamma(s)_j$ $(1 \leq j \leq n)$ in this order.
3. $\gamma(0,\cdot)$ parametrizes the half line $[0,\infty]$.

Take $\gamma(\cdot) := \gamma(1,\cdot)$ Slit $\mathbb{P}^1$ open along γ to obtain a simply connected domain U. Let $\varphi(z;t)$ be a holomorphic branch of

$$\prod_{j=0}^{n+1} (t - z_j)^{-\mu_j}$$

for $t \in U$ and let $S(z;t)$ be holomorphic on U having $\varphi(z;t)$ as its derivative (with respect to t) and such that $S(z;0) = 0$. The mapping $S(z;t)$ resembles the Schwarz-Christoffel mapping. Now by Stokes we have:

$$\mathrm{Vol}(M(z)) = \int_U \Omega(z) = \frac{1}{2i} \int_{\partial U} \overline{S(z)} dS(z)$$

Note that the boundary of U consists of twice the curve γ, once in each direction. Let S_j^+ and S_j^- respectively denote the images under $S(z;t)$ of the points $z_0, \ldots, z_{n+2}$ when we pass from 0 to ∞ along ∂U in positive and negative orientation respectively.

In particular, note that S_j^+ is the j-th vertex $v_j(z)$ (continued along Γ). These numbers satisfy:

1. $S_0^+ = S_0^- = 0$ and $S_{n+2}^+ = S_{n+2}^-$.
2. For all j, $\overline{\omega}_j(S_{j+1}^+ - S_j^+) = \omega_j(S_{j+1}^- - S_j^-)$

Define $w_j := \overline{\omega}_j(S_{j+1}^+ - S_j^+) = \overline{\omega}_j e_j(z)$ for $0 \le j < n+2$. If we take $w_{n+2} := -S_{n+2}^+ = e_{n+2}(z)$ then

$$w := (w_0, \ldots, w_{n+2}) \in \mathrm{Pol}(\mu)$$

Let $S_j^+(t)$ and $S_j^-(t)$ be the branches of $S(z;t)$ on the curve segment $[z_j, z_{j+1}]$ such that $S_j^\pm(z_j) = S_j^\pm$ and $S_j^\pm(z_{j+1}) = S_{j+1}^\pm$. Then for all j there exist $\beta_j \in \mathbb{C}$ such that:

$$S_j^+(t) = \omega_j^2 S_j^-(t) + \overline{\beta}_j$$

Substituting this in the RHS of the Stokes equality yields (integrations are along γ):

$$\mathrm{Vol}(M(z)) = \sum_{j=0}^{n+1} \frac{1}{2i}\left(\int_{z_j}^{z_{j+1}} \overline{S_j^+(t)}dS_j^+(t) - \int_{z_j}^{z_{j+1}} \overline{S_j^-(t)}dS_j^-(t)\right) =$$

$$= \sum_{j=0}^{n+1} \frac{1}{2i}\int_{z_j}^{z_{j+1}} \beta_j \omega_j^2 dS_j^-(t) = \sum_{j=0}^{n+1} \frac{\beta_j}{2i}(S_{j+1}^+ - S_j^+)$$

Hence this volume does depend only on the points S_j^+ $(= v_j(z))$, *not* on the curve connecting them. Replacing the subsequent connecting curves all by straight line segments and recalling the definition of $w \in \mathrm{Pol}(\mu)$ we get:

$$\mathrm{Vol}(M(z)) = \mathrm{Area}(P_+(w)) - \mathrm{Area}(P_-(w)) = H(w,w)$$

This proves the theorem. $\square$

2.4 Geometry and monodromy of F_D

In this section we will assume that the parameters μ_j are all *rational*. The edge functions defined in the previous section are multivalued analytic on $X \subset (\mathbb{P}^1)^n$. They span locally an $(n+1)$-dimensional space over $\mathbb{C}$ at any point of X. By remark 2.1 we conclude that the edges form in fact a local system on X which gives rise to a representation of the fundamental group of X. The complement D of X in $(\mathbb{P}^1)^n$ is the union of a finite number of divisors with equations $z_i = z_j$. The space $(\mathbb{P}^1)^n$ has a natural stratification such that the dense open set X is the highest dimensional stratum. The strata are indexed by partitions Π of $\{0, \ldots, n+2\}$ satisfying

(i) For all $p \in \Pi$: $\#p \le n+1$.

(ii) For all $p \in \Pi$: $\#(p \cap \{0, n+1, n+2\}) \le 1$.

We define a partial ordering on partitions such that $\Pi_1 \leq \Pi_2$ iff Π_2 is a refinement of Π_1. The stratum D_Π for such a partition is defined by

$$D_\Pi = \{(z_1, \ldots, z_n) \in \mathbb{P}^1(\mathbb{C})^n \mid z_i = z_j \text{ iff } \{i,j\} \subseteq p \text{ for some } p \in \Pi\}$$

where i and j range over $\{0, \ldots, n+2\}$. Then the dimension of D_Π in $(\mathbb{P}^1)^n$ is $\#\Pi - 3$. Note that X is the stratum corresponding to the partitioning in singletons and that D_Π contains $D_{\Pi'}$ in its closure iff $\Pi' \leq \Pi$. The edge function is really a function F of Nilsson class on $(\mathbb{P}^1)^n$ of determination order $n+1$ and singularities along D. However, to study the function F it will be useful to embed X in a different n-dimensional space, Q, endowed with a stratification such that X is the stratum of highest dimension. The strata are indexed by partitions Π of $\{0, \ldots, n+2\}$ satisfying

(i) For all $p \in \Pi$: $\#p \leq n+1$.

(ii) For all $p \in \Pi$: $\sum_{j \in p} \mu_j < 1$.

We call such partitions *μ-stable* or just *stable*. The stratum D_Π will again be of dimension $\#\Pi - 3$. We construct Q by using Geometric Invariant Theory of Hilbert-Deligne-Mumford [MF]. Let $N \in \mathbb{N}$ be the smallest common denominator of all μ_j and set $m_j = N\mu_j$. Let for any $m \in \mathbb{Z}$, $\mathcal{O}(m)$ denote the line bundle of degree m over $\mathbb{P}^1$. If $m \geq 0$, we can interpret sections in this bundle as homogeneous polynomials of degree m on $\mathbb{C}^2$, where $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$. Let the line bundle $\mathcal{L}$ over $(\mathbb{P}^1)^{n+3}$ be defined as the exterior tensor product:

$$\mathcal{L} = \bigotimes_{j=0}^{n+2} \mathcal{O}(m_j).$$

Now $\mathrm{PGL}(\mathbb{C}^2)$ acts by the diagonal action on $(\mathbb{P}^1)^{n+3}$ and because $\sum_j m_j$ is even, $\mathcal{L}$ admits a unique structure of a homogeneous $\mathrm{PGL}(\mathbb{C}^2)$ bundle. With respect to $\mathcal{L}$, the semi-stable (resp. stable) points of $(\mathbb{P}^1)^{n+3}$ are given by

$$\{(z_0, \ldots, z_{n+2}) \mid \text{for all } j, \sum_{z_i = z_j} \mu_i \leq 1 \text{ (resp. } < 1)\}$$

Now we take Q as the geometric quotient:

$$Q = (\mathbb{P}^1)^{n+3}_{stable}/\mathrm{PGL}(\mathbb{C}^2)$$

The space Q is a smooth (quasi projective) variety (e.g. see [DO, chap. 2 Thm. 2]). We embed X in Q by

$$(z_1, \ldots, z_n) \mapsto \text{orbit of } (0, z_1, \ldots, z_n, 1, \infty).$$

For any stable partition Π, define the stratum D_Π by

$$\{(z_0, \ldots, z_{n+2}) \mid z_i = z_j \text{ iff } \{i,j\} \subseteq p \text{ for some } p \in \Pi\}/\mathrm{PGL}(\mathbb{C}^2)$$

This defines a stratification of Q as indicated. If $\mu_i + \mu_j < 1$ for some $i \neq j$ we denote the $(n-1)$-dimensional stratum D_Π where Π is the maximal partition containing $\{i,j\}$ by $[i\ j]$.

By identifying X and its embedding in Q, we can view the Nilsson class function F as a Nilsson class function on Q with singularities along the boundary of X. Let $U \subset X \subset Q$ be a small simply connected neighborhood of $z^o \in X$. Let $V = V(U)$ denote the $\mathbb{C}$-vectorspace spanned by all determinations of F on U. Then by previously obtained results, V is $(n+1)$-dimensional. By analytic continuation we get a natural *right* representation of the fundamental group of X on V:

$$M : \pi_1(X, z^o) \to \mathrm{GL}(V)$$

We call this the *monodromy representation.* There is a canonical mapping of U into the dual V' of V, the *evaluation mapping.* It is given by:

$$ev : U \to V', \ ev : z \mapsto \text{evaluation at } z$$

Note that ev can be continued analytically throughout X. Henceforth we will view ev as a multivalued holomorphic mapping of X into V'. We want to understand the behaviour of this evaluation mapping, or in fact its projective version:

$$\mathrm{pev} : X \to \mathbb{P}(V')$$

In particular we want to study when (i.e. for which μ) this mapping has a *single valued* holomorphic inverse on its image. If such an inverse exists, this implies that monodromy induces a *discrete* group in $\mathrm{PGL}(V')$. The idea is to study *local* properties of pev first and use the results to understand the *global* properties.

Theorem 2.4 *The space V' admits a hyperbolic hermitean form H, invariant under dual monodromy (the transpose of M, i.e. a* left *representation). Moreover, evaluation maps X into the positive part of V' (with respect to H).*

Proof: Let w again be the (multivalued) mapping

$$w(z) := (\overline{\omega}_0 e_0(z), \overline{\omega}_1 e_1(z), \ldots, \overline{\omega}_{n+2} e_{n+2}(z))$$

Then $w(z) \in Pol(\mu)$. Because $e_0, \ldots, e_n$ span V, by theorem 2.2 there exists a hyperbolic hermitean matrix $H \in \mathrm{Mat}(n+1, \mathbb{C})$ such that

$$H(w(z), w(z)) = \sum_{0 \leq i,j \leq n} H_{ij} e_i(z) \overline{e_j(z)}$$

for all $z \in X$. Because the left hand side of this equality equals $\mathrm{Vol}(M(z))$ the right hand side is invariant under monodromy. Let $\lambda_0, \ldots, \lambda_n$ be the dual basis of V' with respect to $e_0, \ldots, e_n$. Define $H(\lambda_i, \lambda_j) := H_{ij}$. This is an invariant hyperbolic hermitean form and:

$$H(\mathrm{ev}(z), \mathrm{ev}(z)) = H(\sum_i e_i(z)\lambda_i, \sum_j e_j(z)\lambda_j) =$$

$$= \sum_{i,j} H_{ij} e_i(z)\overline{e_j(z)} = \mathrm{Vol}(M(z)) > 0$$

This proves the theorem. □

The subspace B of $\mathbb{P}(V')$ given by

$$B := \{[v] \mid H(v,v) > 0\}$$

is isomorphic to the complex unit ball in $\mathbb{C}^n$. By the previous theorem we conclude that pev maps X into B.

Theorem 2.5 *The mapping pev is everywhere locally biholomorphic.*

Proof: For $y \in X$ let $f_0, \ldots, f_n$ be a local basis of determinations and $y_1, \ldots, y_n$ some local coordinates. Then pev is locally biholomorphic at y iff the following *wronskian* does not vanish near y:

$$\det \begin{pmatrix} f_0 & \frac{\partial f_0}{\partial y_1} & \cdots & \frac{\partial f_0}{\partial y_n} \\ f_1 & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ f_n & \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

Now by remark 2.1 every determination near y is completely determined by its value and those of its first order partial derivatives at y. This clearly implies that the wronskian does not vanish at y. □

To examine local behavior along the singular locus, we extend the integral representation of the edges to

$$\{(z_0, \ldots, z_{n+1}, \infty) \in (\mathbb{P}^1)^{n+3} \mid \#\{z_0, \ldots, z_{n+1}, \infty\} = n+3\}$$

by the formula

$$E_{ij} := (z_{n+1} - z_0)^{1-\mu_{n+2}} \int_{z_i}^{z_j} \prod_{k=0}^{n+1} (s - z_k)^{-\mu_k} ds.$$

Here integration is along any path avoiding (except in its end points) all z_k. One computes

$$E_{ij} = \int t^{-\mu_0}(t-1)^{-\mu_{n+1}} \prod_{k=1}^{n} (t - \frac{z_k - z_0}{z_{n+1} - z_0})^{-\mu_k} dt$$

(integrate along the transformed path) so E is just an extension of the edges invariant under the stabilizer of ∞ (linear transformations). From this integral representation one deduces the following important lemma.

Lemma 2.2 *Let $J \subset \{0, 1, \ldots, n+2\}$ be such that $2 \leq \#J \leq n+1$ and $\Sigma_J := \sum_{j \in J} \mu_j < 1$. Let Π denote the maximal stable partition containing J. Then the Nilsson class function F only has two different exponents along the stratum D_Π. The several possibilities are listed below with their respective multiplicities.*

J satisfies:	$n+2-\#J$ *times*	$\#J-1$ *times*
$\{0, n+1\} \not\subseteq J \subset \{0, \ldots, n+1\}$	0	$1 - \Sigma_J$
$\{0, n+1\} \subseteq J \subset \{0, \ldots, n+1\}$	$1 - \mu_{n+2}$	$2 - \Sigma_J - \mu_{n+2}$
$n+2 \in J \subset \{1, \ldots, n, n+2\}$	$\Sigma_J - \mu_{n+2}$	$1 - \mu_{n+2}$
None of the above	$\Sigma_J - 1$	0

Proof: Compute this from the extended integral representation of the edges. □

Corollary 2.1 *For a stratum as in the previous lemma and $q \in D_\Pi$, the limit $\lim_{z \to q} pev(z)$ exists and does not depend on local monodromy near q. A small neighborhood of q intersected with D_Π will be mapped into a subspace of dimension $\#J - 1$ of $\mathbb{P}(V')$ by this limiting process.*

Proof: Because only evaluation upto some scalar multiple is considered, the exponents along D_Π can be shifted to obtain an exponent 0 with multiplicity $n+2-\#J$ and an exponent $1 - \Sigma_J$ with multiplicity $\#J - 1$. The corollary now follows from the fact that $1 - \Sigma_J > 0$.□

For strata of codimension one we need the following stronger result.

Theorem 2.6 *Let $q \in [i\ j]$ for some $(n-1)$-dimensional stratum $[i\ j]$ of Q. There exists a neighborhood Q_q of q, holomorphic functions $q_0, \ldots, q_n$ on Q_q and homogeneous coordinates on $\mathbb{P}(V')$ such that*

(i) The set $[q_0 = 0]$ equals $[i\ j] \cap Q_q$.

(ii) The mapping $Q_q \to \mathbb{P}(V')$, $w \mapsto (q_0(w) : \ldots : q_n(w))$ is biholomorphic.

(iii) The projective evaluation mapping pev on Q_q is just

$$(q_0 : q_1 : \ldots : q_n) \mapsto (q_0^{1-\mu_i-\mu_j} : q_1 : \ldots : q_n)$$

Proof: According to lemma 2.2 there exists a coordinate neighborhood

$$(Q_q, w_1, \ldots, w_n)$$

of q and holomorphic functions $f_0, \ldots, f_n$ on Q_q such that

(i) The set $[w_1 = 0]$ equals $[i\ j] \cap Q_q$.

(ii) At each point of Q_q the functions

$$f_0 \cdot w_1^{\alpha}, f_1 \cdot w_1^{\beta}, \ldots, f_n \cdot w_1^{\beta}$$

form a basis of determinations of F. Here α and β are the two exponents along $[i\ j]$.

Then with respect to suitable homogeneous coordinates of $\mathbb{P}(V')$ evaluation on Q_q is:

$$\text{pev} : w \mapsto (w_1^{1-\mu_i-\mu_j} f_0 : f_1 : \ldots : f_n)$$

Now it is well known that the Wronskian of $f_0, \ldots, f_n$ with respect to $w_1, \ldots, w_n$ satisfies a first order system of linear differential equations and from the explicitly known equations for the Lauricella function one deduces that the Wronskian has the following form near q:

$$h \cdot w_1^{\alpha+n\beta-1}$$

Here h is a holomorphic function which does not vanish at q. Explicitly computing this wronskian using Cramer's rule, we find that both f_0 and

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_n} \\ \frac{\partial f_2}{\partial w_1} & \cdots & \frac{\partial f_2}{\partial w_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial w_1} & \cdots & \frac{\partial f_n}{\partial w_n} \end{pmatrix}$$

do not vanish at q. By taking Q_q small enough, the functions

$$q_0 := w_1 \cdot f_0^{1/(1-\mu_i-\mu_j)},\ q_1 := f_1, \ldots,\ q_n := f_n$$

satisfy the conditions of the theorem. $\square$

2.5 Ramified coverings of Q

In this section we will prove the following main result of these notes:

Theorem 2.7 *Suppose that for all strata $[i\ j]$ of Q the exponent difference $1-\mu_i-\mu_j$ along $[i\ j]$ equals $1/m_{ij}$ for some $m_{ij} \in \mathbb{N}_{\geq 2}$. Then the image of the projective evaluation mapping is dense in B and there exists a* single valued *holomorphic mapping $\Phi : B \to Q$ such that on X one has $\Phi \circ pev = id_X$.*

We will prove this result by studying ramified coverings of Q. A crucial ingredient of the proof is the existence of a monodromy invariant metric d on B generating its topology. This is the so-called *Poincaré-Bergman metric* defined as follows.

Definition 2.2 *Define a metric d on B by:*

$$cosh\, d([v_1],[v_2]) = \frac{|H(v_1,v_2)|}{[H(v_1,v_1)H(v_2,v_2)]^{1/2}}$$

For any $\epsilon > 0$ denote the ball of radius ϵ centered at $b \in B$ by $B(\epsilon, b)$:

$$B(\epsilon,b) := \{b' \in B \mid d(b',b) < \epsilon\}$$

This metric is clearly monodromy invariant, and it generates the topology of B. Let $\pi : \widetilde{X} \to X$ be the universal covering of X. Lift pev to a (single-valued) locally biholomorphic map $\overline{\text{pev}}$ on $\widetilde{X}$. Then $\text{Aut}(\widetilde{X}|X)$ is isomorphic to $\pi_1(X, z^o)$.

Theorem 2.8 *Suppose $q \in D_\Pi$ and for all strata $[i\ j]$ containing q in their closure the exponent difference $1-\mu_i-\mu_j$ equals $1/m_{ij}$ for some $m_{ij} \in \mathbb{N}_{\geq 2}$. Let $\pi^*(\Pi) : X^*(\Pi) \to X$ be the universally ramified covering of X ramified of order m_{ij} along $[i\ j]$. So any m_{ij}-fold loop around $[i\ j]$ induces the identity automorphism of $X^*(\Pi)$ and $X^*(\Pi)$ is universal with respect to this property. Then the covering $X^*(\Pi)$ embeds in a ramified covering*

$$\pi(\Pi) : X(\Pi) \to \bigcup_{\Pi' \geq \Pi} D_{\Pi'}$$

i.e. $X^*(\Pi)$ is a submanifold of $X(\Pi)$ and $\pi^*(\Pi)$ is the restriction of $\pi(\Pi)$ to $X^*(\Pi)$. Moreover, $\overline{pev}$ induces a locally biholomorphic mapping on $X(\Pi)$, also denoted by $\overline{pev}$.

Proof: From theorem 2.6 it follows that the evaluation mapping is invariant under analytic continuation along any m_{ij}-fold loop around $[i\ j]$. Hence $\overline{\text{pev}}$

descends to a locally biholomorphic mapping $\overline{\mathrm{pev}}$ on $X^*(\Pi)$. The proof now proceeds by induction on the dimension n. In dimension one this embedding of coverings is just the remark that

$$\mathbb{C}^* \stackrel{z^m}{\to} \mathbb{C}^*$$

extends to a mapping of $\mathbb{C}$ onto $\mathbb{C}$. Let $w \in D_{\Pi'}$ for some stable $\Pi' \geq \Pi$. Let $p_1, \ldots, p_s \in \Pi'$ be the parts containing at least two elements. Then there are local coordinates

$$w_1^0, \ldots, w_{\#\Pi'-3}^0, w_1^1, \ldots, w_{\#p_1-1}^1, w_1^2, \ldots, w_{\#p_2-1}^2, \ldots, w_{\#p_s-1}^s$$

on the polydisc $|w_j^m| < 1$ centered at w such that the strata in this polydisc are described as the intersection structure of the hyper planes:

$$\sum_{k \leq j \leq l} w_j^m = 0$$

Here m ranges over $\{1, \ldots, s\}$ and and k, l over all values such that $1 \leq k \leq l \leq \#p_m - 1$. Let Δ^m denote the m^{th} "coordinate slice", i.e. the set of points of which only the $w_\cdot^m$ coordinates are non-zero. Then the polydisc neighborhood of w is a product

$$\Pi_{j=0}^s \Delta^j$$

Moreover, the stratification is compatible with this product, i.e. strata are products of their projections on the coordinate slices. Every slice Δ^p with its stratification is isomorphic to a polydisc neighborhood on some geometric quotient Q_p of dimension $\#p - 1$ (including stratifications). For example take μ_j for $j \in p$ and twice $1 - \frac{1}{2}\sum_{j \in p} \mu_j$ as the new $\#p + 2$ parameters.

If no set in Π has $n+1$ elements, then for all points w as above the factors of such products have lower dimension than Q. This allows an inductive procedure in this case. Let U be a polydisc neighborhood of w as before, then the universally ramified covering of $U \cap X$ embeds in a ramified covering of U, and $\overline{\mathrm{pev}}$ extends locally biholomorphically over this ramified covering. (Because $U \cap X$ is just a product of lower dimensional situations). The only automorphism of this local ramified covering over U that fixes $\overline{\mathrm{pev}}$ is the trivial automorphism because $\overline{\mathrm{pev}}$ is locally biholomorphic everywhere and the pre-image of w in the covering is fixed by any automorphism. This implies that the quotient map of this universally ramified covering of $U \cap X$ to the covering $X^*(\Pi)$ is actually an *embedding*. So all local extensions fit together and we conclude that $X^*(\Pi)$ embeds in a covering $X(\Pi)$ as stated.

By theorem 2.6 the evaluation mapping extends to a locally biholomorphic mapping on all points of $X(\Pi)$ above co-dimension one strata. Then by Hartog's theorem the evaluation mapping extends locally biholomorphically to all of $X(\Pi)$.

The case remains that Π contains a set of $n+1$ elements (i.e. $\{q\}$ is itself a stratum). By reasoning in the same way as before, we conclude that $X^*(\Pi)$ embeds in a ramified covering

$$\pi^\times : X^\times(\Pi) \to \bigcup_{\Pi' > \Pi} D_{\Pi'}$$

and evaluation extends locally biholomorphically to this covering. We have to show that it extends over the point q. Let Q_q be a small ball neighborhood of q and $Q^\times$ a connected component of $(\pi^\times)^{-1}(Q_q)$.

By corollary 2.1 on $Q^\times$ the limit $\lim_{\pi^\times(w)\to q} \overline{\mathrm{pev}}(w) =: b$ is a well defined point in B, in particular, it is fixed by local monodromy near q. Let $K \subset Q_q$ be a compact ball around q such that for any $w \in Q^\times \cap (\pi^\times)^{-1}(\partial K)$ the distance $d(\overline{\mathrm{pev}}(w), b)$ is at least $2\delta > 0$. Such a K exists because $\overline{\mathrm{pev}}$ is locally biholomorphic and this distance is invariant under the automorphisms of $Q^\times \mid Q_q$. We will show that $\overline{\mathrm{pev}}$ maps some open subset of $K^\times := (\pi^\times)^{-1}(K)$ biholomorphically onto the punctured ball $B(\delta, b) - \{b\}$. Then by Hartog $\pi^\times \circ (\overline{\mathrm{pev}})^{-1}$ extends over b which shows that $X^*(\Pi)$ indeed embeds in a covering $X(\Pi)$ as stated.

Take a covering sequence of compact subsets of $K - \{q\}$

$$K_1 \subset\subset K_2 \subset\subset \dots$$

i.e. for all j, K_j is contained in the *interior* of K_{j+1} and $\cup\{K_j \mid j \geq 1\} = K - \{q\}$. If $\epsilon > 0$ a point $w \in Q^\times$ will be called *ϵ-wide* if $\overline{\mathrm{pev}}$ maps some neighborhood of w biholomorphically onto the ball $B(\epsilon, \overline{\mathrm{pev}}(w))$.

Lemma 2.3 *For each $j \geq 1$ there exists an $\epsilon_j > 0$ such that any point $w \in K_j^\times := (\pi^\times)^{-1}(K_j)$ is ϵ_j-wide.*

Proof: Consider for all $N \geq 1$ the set

$$W_N = \{w \in \ \mid w \text{ is } \eta\text{-wide for some } \eta > 1/n\}.$$

The following properties hold for these sets:

(i) W_N is an open set for all N.

(ii) If $N \geq M$ then $W_N \supseteq W_M$.

(iii) Each W_N is $\mathrm{Aut}(Q^\times | Q_q)$ stable.

(iv) $\cup\{W_N \mid N \geq 1\} = Q^\times$.

Property (iii) follows by invariance of the distance d on B and (iv) follows because $\overline{\mathrm{pev}}$ is locally biholomorphic. Now all K_j are compact and hence there exist integers $1 \le N_1 \le N_2 \le \ldots$ such that $K_j^\times \subset W_{N_j}$ for all j. This implies that all $w \in K_j^\times$ are $1/N_j$-wide. $\Box$

Let $b_o = \overline{\mathrm{pev}}(w)$ be in $B(\delta, b)$ for some $w \in Q^\times$. Locally near b_o the mapping $\overline{\mathrm{pev}}$ has a holomorphic inverse ψ. Let $\gamma : [0,1] \to B(\delta, b) - \{b\}$ be any curve in the punctured ball such that $\gamma(0) = b_o$. Suppose that ψ can be continued analytically along γ upto (but not necessarily including) $\gamma(t)$ for some $t \in (0,1]$. Then ψ maps into $K^\times$ because its image cannot cross $\partial K^\times$. If $\psi \circ \gamma(t') \in K_j^\times$ for some $t' \in (0,t)$ such that $d(\gamma(t'), \gamma(t)) \le \epsilon_j$ then by the wideness lemma, ψ can be continued upto *and including* $\gamma(t)$. Now this is always the case for the only other possibility is that $\pi^\times \circ \psi \circ \gamma$ tends to q if t' tends to t. But this would imply that $\gamma(t) = b$ which we assumed not to be the case.

Now b is of co-dimension at least two in $B(\delta, b)$ so $\pi^\times \circ \psi$ extends to a holomorphic mapping on $B(\delta, b)$ by simply-connectedness of $B(\delta, b) - \{b\}$ and Hartog's theorem. This mapping extends the covering $\pi^\times$ over q, proving the existence of an embedding of $X^*(\Pi)$ in $X(\Pi)$ as stated and $\overline{\mathrm{pev}}$ extends locally biholomorphically. $\Box$(theorem 2.8)

The main theorem follows from theorem 2.8:

Proof: (Of theorem 2.7). Suppose that for all strata $[i\ j]$ the exponent difference $1 - \mu_i - \mu_j$ equals $1/m_{ij}$ for some $m_{ij} \in \mathbb{N}_{\ge 2}$. Let $\pi^*(m) : X^*(m) \to X$ be the universally ramified covering with ramification order m_{ij} along $[i\ j]$. Then $\overline{\mathrm{pev}}$ descends to a locally biholomorphic mapping $\overline{\mathrm{pev}}$ on $X^*(m)$ because it is invariant under continuation along any m_{ij} fold loop around $[i\ j]$.

For any $q \in D_\Pi$ a connected component of $(\pi^*)^{-1}(Q_q)$ for some neighborhood Q_q is isomorphic to a connected component of $X^*(\Pi)$ over Q_q because $\overline{\mathrm{pev}}$ is invariant under the trivial automorphism of such a component only (by theorem 2.8). This implies that $X^*(m)$ embeds in a ramified covering $\pi(m) : X(m) \to Q$ in the same sense as before.

The (quasi projective) variety Q has a natural (projective) compactification $\overline{Q}$ (the *universal categorical quotient* $(\mathbb{P}^1)^{n+3}_{semistable}/\mathrm{PGL}(\mathbb{C}^2)$). The complement $\overline{Q} - Q$ consists of a finite number of (singular) points and if $q \to \overline{Q} - Q$ then $\mathrm{pev}(q)$ will tend arbitrarily far away from any point in B (with respect to the metric d). One can compute this simply using the integral representations of edges or see the discussion in [DM]. Now a wideness argument applied to $X(m) \mid Q$ as before shows that any local holomorphic inverse of $\overline{\mathrm{pev}}$ extends to a global inverse on B. Moreover $\overline{\mathrm{pev}}$ establishes an isomorphism between $(X(m), \mathrm{Aut}(X(m)|Q))$ and $(B, M(\pi_1(X, z^o)))$.$\Box$

2.6 Some additional results

The main theorem discussed here is not the end of the story. Suppose that for some i, j equality $\mu_i = \mu_j$ holds. Then interchanging coordinates i and j on $(\mathbb{P}^1)^{n+3}$ (numbered $0, \ldots, n+2$) induces a transformation of Q. The Nilsson class function F of edges is invariant under this transformation. So we have a subgroup Σ of the symmetric group S_{n+3} acting on Q and stabilizing F. This allows one to consider the induced system F on the quotient $\Sigma\backslash Q$. The upshot of this is that the exponent along a corresponding stratum $[i\ j]$ will be $(1 - \mu_i - \mu_j)/2$ and in fact the more general theorem becomes:

Theorem 2.9 *Suppose that for all strata $[i\ j]$ the exponent difference $1-\mu_i-\mu_j$ is either $1/m_{ij}$ (if $\mu_i \neq \mu_j$) or $2/m_{ij}$ (if $\mu_i = \mu_j$) then the evaluation mapping pev on $\Sigma\backslash Q$ has a holomorphic inverse on B.*

Unfortunately, this quotient will in general be singular because Σ does not even have to act free on X. So one has to take care of additional details to deal with this, essentially without changing the idea of the proof of such a theorem (see [M]). In fact, would this quotient be smooth, then the same proof as discussed in these notes would apply. I omitted this additional theory for the sake of keeping things more transparant.

A second important remark is that one can replace the condition that $\mu_j \in (0,1)$ for all j by the condition $\mu_j > 0$ for all j. This would add the *elliptic* and *parabolic* cases to our theory (if for some j, $\mu_j > 1$ or $\mu_j = 1$ respectively). One can again prove the above theorem for these cases. (Now Q will just be projective space $\mathbb{P}^n$). Though in addition to discussing symmetries Σ, one has to do some extra work to infer invariant forms for the monodromy (which will be *definite* and *semi-definite* respectively).

In the *elliptic* case, the monodromy is *finite*, implying that Lauricella's F_D is *algebraic*. A holomorphic inverse for $\overline{\text{pev}}$ then exists throughout $\mathbb{P}(V')$. In the *parabolic* (sometimes called *euclidean*) case monodromy acts by *affine* transformations. A holomorphic inverse for $\overline{\text{pev}}$ then exists on a affine space in $\mathbb{P}(V')$. In this parabolic case the constant functions always satisfy the equations of F_D! Some work was done on these elliptic and parabolic cases, though by different means. The question investigated is if monodromy is *discrete*, not if an inverse of the evaluation exists. For example see [Sa], [CW]. Cohen and Wolfart use arithmetic properties of monodromy to deduce finiteness or discreteness (in the euclidean case).

It is an interesting remark that in the parabolic cases, the quotient $\Sigma\backslash Q$ is always a weighted projective space. Hence some positive results are obtained for the conjecture in [BS] that the quotient of an affine space with respect to a discrete cocompact action of a group generated by reflections will always be weighted projective space. The weights are essentially just the degrees of the irreducible factors of Σ. Here is a list of the parabolic cases:

n	denominator	numerators	weights
2	4	4 1 1 1 1	2 3 4
	6	6 2 2 1 1	1 2 2
	6	6 3 1 1 1	1 2 3
3	6	6 2 1 1 1 1	1 2 3 4
4	6	6 1 1 1 1 1 1	2 3 4 5 6

The next section shows a list of all 102 cases in which evaluation has a globally holomorphic inverse.

2.7 Tables

$n = 2$

#	den.	Numerators				
1	3	2	1	1	1	1
2	4	2	2	2	1	1
3	4	3	2	1	1	1
4	4	4	1	1	1	1
5	5	2	2	2	2	2
6	6	3	3	2	2	2
7	6	3	3	3	2	1
8	6	4	3	2	2	1
9	6	4	3	3	1	1
10	6	4	4	2	1	1
11	6	5	2	2	2	1
12	6	5	3	2	1	1
13	6	5	4	1	1	1
14	6	6	2	2	1	1
15	6	6	3	1	1	1
16	6	7	2	1	1	1
17	6	8	1	1	1	1
18	8	4	3	3	3	3
19	8	5	5	2	2	2
20	8	6	3	3	3	1
21	9	4	4	4	4	2
22	10	6	5	3	3	3
23	10	6	6	3	3	2
24	10	7	4	4	4	1
25	10	8	3	3	3	3
26	10	9	3	3	3	2
27	12	5	5	5	5	4
28	12	6	5	5	4	4
29	12	6	5	5	5	3
30	12	7	5	4	4	4
31	12	7	6	5	3	3
32	12	7	7	4	4	2
33	12	7	7	6	2	2
34	12	8	5	5	3	3
35	12	8	5	5	5	1
36	12	8	7	3	3	3
37	12	9	7	4	2	2
38	12	9	9	2	2	2
39	12	10	5	3	3	3
40	12	11	7	2	2	2

41	14	8	5	5	5	5
42	14	11	5	5	5	2
43	15	8	6	6	6	4
44	18	8	7	7	7	7
45	18	10	7	7	7	5
46	18	10	10	7	7	2
47	18	11	8	8	8	1
48	18	13	7	7	7	2
49	18	14	13	3	3	3
50	20	10	9	9	6	6
51	20	13	9	6	6	6
52	20	14	11	5	5	5
53	24	14	9	9	9	7
54	24	19	17	4	4	4
55	30	22	11	9	9	9
56	30	23	22	5	5	5
57	30	26	19	5	5	5
58	42	26	15	15	15	13
59	42	34	29	7	7	7

$n = 3$

#	den.	Numerators					
1	3	1	1	1	1	1	1
2	4	2	2	1	1	1	1
3	4	3	1	1	1	1	1
4	6	3	2	2	2	2	1
5	6	3	3	2	2	1	1
6	6	3	3	3	1	1	1
7	6	4	2	2	2	1	1
8	6	4	3	2	1	1	1
9	6	4	4	1	1	1	1
10	6	5	2	2	1	1	1
11	6	5	3	1	1	1	1
12	6	6	2	1	1	1	1
13	6	7	1	1	1	1	1
14	8	3	3	3	3	3	1
15	10	5	3	3	3	3	3
16	10	6	3	3	3	3	2
17	12	5	5	5	3	3	3
18	12	7	5	3	3	3	3
19	12	7	7	4	2	2	2
20	12	9	7	2	2	2	2

$n = 4$

#	den.	Numerators						
1	4	2	1	1	1	1	1	1
2	6	2	2	2	2	2	1	1
3	6	3	2	2	2	1	1	1
4	6	3	3	2	1	1	1	1
5	6	4	2	2	1	1	1	1
6	6	4	3	1	1	1	1	1
7	6	5	2	1	1	1	1	1
8	6	6	1	1	1	1	1	1
9	10	3	3	3	3	3	3	2
10	12	7	7	2	2	2	2	2

$n = 5$

#	den.	Numerators							
1	4	1	1	1	1	1	1	1	1
2	6	2	2	2	2	1	1	1	1
3	6	3	2	2	1	1	1	1	1
4	6	3	3	1	1	1	1	1	1
5	6	4	2	1	1	1	1	1	1
6	6	5	1	1	1	1	1	1	1

$n = 6$

#	den.	Numerators								
1	6	2	2	2	1	1	1	1	1	1
2	6	3	2	1	1	1	1	1	1	1
3	6	4	1	1	1	1	1	1	1	1

$n = 7$

#	den.	Numerators									
1	6	2	2	1	1	1	1	1	1	1	1
2	6	3	1	1	1	1	1	1	1	1	1

$n = 8$

#	den.	Numerators										
1	6	2	1	1	1	1	1	1	1	1	1	1

$n = 9$

#	den.	Numerators											
1	6	1	1	1	1	1	1	1	1	1	1	1	1

2.8 Literature

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Chapter 3

Reflection groups

3.1 Introduction

The investigations that lead to the results in this chapter were mainly motivated by the following three things:

1. The intriguing paper by Orlik and Solomon [OS] in which they study (using a computer) the invariants of Shephard groups. They show that the generating homogeneous invariants can be chosen in such a way that their discriminant is the same as that of a related *real* reflection group.

2. The paper by Deligne and Mostow [DM]. In this paper they construct groups of transformations of a complex ball generated by reflections that act discretely. These groups arise as a *monodromy* group of a hypergeometric function in several variables (a Lauricella F_D).

3. The work of Heckman and Opdam on hypergeometric functions and Bessel functions associated to (crystallographic) root systems as in [H], [O] and other papers.

The goal was to understand the results of Orlik and Solomon in an intrinsic way as follows. Start with the complement of a discriminant of a finite real reflection group W and try to construct the complex groups as monodromy groups of certain special functions associated to the root system of W.

This turns out to be a productive idea. The complex groups arise this way by "altering" the orders of the generating reflections of W. We call these groups *truncated braid groups*. These fall into three categories: the *finite*, the *parabolic* and the *hyperbolic* groups. For each of these categories the results include:

1. Geometric information about ramified coverings of discriminant complements.
2. Presentations for the complex (not necessarily finite) groups.
3. Chevalley like theorems on the invariants of these groups.

The results of [OS] and of Coxeter [C] (on presentations of finite complex reflection groups) are consequences of the theory for the finite case.

The results in the parabolic case can be related to results of Looijenga [L] and Bernstein Schwarzman [BS]. In [BS] it is conjectured that if a group generated by complex reflections of an affine space acts discretely and cocompactly the quotient space is always weighted projective. Indeed, in our examples the quotient is weighted projective and the weights are directly related to the degrees of the real Coxeter group. As in [L] a Chevalley like theorem is proved for certain rings of theta functions.

The hyperbolic case gives more examples of discrete groups acting on a complex ball of which the quotient (and other things) can be described explicitly. Two remarks should be made. Firstly there is a non zero intersection between this paper and [DM]. The theory for *classical* root systems can be translated to the theory of Lauricella's F_D. Details will appear in a seperate article. Secondly, at the moment not all hyperbolic cases are treated in all detail. For several groups the ball quotient will no longer be a weighted projective space. The algebraic construction of these quotients similar to Geometric Invariant Theory is only sketchy on some points. Results upto this point are discussed in the next chapter.

The results for $n = 2$ are (more or less) analogous to those of Milnor in [N] on covering spaces of Pham-Brieskorn varieties.

The rough plans for the development of the theory were laid out by G. Heckman. I would like to thank him for his enormous support. I would also like to thank E. Looijenga, J. Steenbrink and H. de Vries for several interesting discussions and reading of the manuscript.

3.2 Coxeter groups, braid groups and reflection representations

First we introduce some concepts from the theory of root systems and reflection groups Let $(E, (\cdot,\cdot))$ be an Euclidean vector space, $\dim(E) = n$. Let V be its complexification, $V = \mathbb{C} \otimes E$. Extend $(\cdot,\cdot)$ to a bilinear form on V. Let $R \subseteq E$ be a normalized rootsystem of full rank, i.e. a finite set such that:

1. $(\alpha, \alpha) = 2$ for all $\alpha \in R$.

2. $s_\alpha(\beta) := \beta - (\beta, \alpha)\alpha \in R$ for all $\alpha, \beta \in R$.

3. $\mathrm{Span}_{\mathbb{R}}(R) = E$.

If in addition the following holds

4. If $R = R_1 \cup R_2$ and $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$, then $R_1 = \emptyset$ or $R_2 = \emptyset$

then we call R *irreducible*. For any $\alpha \in R$ we denote its dual in V^* by α^*, i.e. $\alpha^*(v) = (\alpha, v)$ for any $v \in V$. Define the *regular* points in V by:

$$V^{reg} = \{v \in V \mid (\alpha, v) \neq 0, \text{ all } \alpha \in R\}$$

Take a set of positive roots R^+, $R = R^+ \cup -R^+$ and simple roots $\alpha_1, \ldots, \alpha_n \in R^+$. Denote the positive chamber in E by E^+:

$$E^+ = \{v \in E \mid (v, \alpha_i) > 0 \text{ for all } i \in \{1, \ldots, n\}\}$$

Denote the group generated by all reflections s_α, $\alpha \in R$ by W:

$$W = \langle s_\alpha \mid \alpha \in R \rangle = \langle s_{\alpha_1}, \ldots, s_{\alpha_n} \rangle$$

Denote the fundamental "weights" by $\lambda_1, \ldots, \lambda_n$, i.e. $(\lambda_i, \alpha_j) = \delta_{ij}$. Define the Coxeter integers m_{ij} by:

$$m_{ij} = \mathrm{order}(s_{\alpha_i} s_{\alpha_j})$$

Then for $i \neq j$:

$$(\alpha_i, \alpha_j) = -2\cos(\frac{\pi}{m_{ij}})$$

We denote the Coxeter element $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ of W by c, and the Coxeter number of W by h, i.e. $\mathrm{order}(c) = h$. If R is irreducible the exponents of W are written $m_1, \ldots, m_n$.

The matrix $M = (m_{ij})$ is called the *Coxeter matrix* of R. Such a matrix can be denoted graphically as follows. Take n vertices $v_1, \ldots, v_n$. Whenever $m_{ij} > 2$ for some $i \neq j$, connect v_i and v_j by an edge, moreover, if $m_{ij} > 3$ write this number along the edge. We will identify the diagram and the matrix, so we can speak of a Coxeter diagram M with Coxeter integers m_{ij}, etc. Next we introduce a braid group associated to M.

Definition 3.1 *For a, b in some group or algebra and $m \in \mathbb{N}$, we define $(a, b)^m$ by:*

$$(a, b)^m = (ab)^{\frac{m}{2}} \quad \text{if } m \text{ is even}$$

$$(a, b)^m = (a, b)^{m-1} \cdot a \quad \text{if } m \text{ is odd}$$

The braid group $B(M)$ associated with the Coxeter diagram M is the group defined by generators and relations as follows:

$$B(M) = \langle g_1, \ldots, g_n | (g_i, g_j)^{m_{ij}} = (g_j, g_i)^{m_{ji}}, \ 1 \leq i < j \leq n \rangle$$

The element $g_1 g_2 \cdots g_n$ is called a Coxeter element *in $B(M)$.*

Lemma 3.1 *Take $i, j \in \{1, \ldots, n\}$ and $i \neq j$. The following statements are equivalent:*

1. *The two generators g_i and g_j in $B(M)$ are conjugate.*
2. *The vertices v_i and v_j are still connected in the diagram M if we erase all edges along which there is an even number.*
3. *The simple roots α_i and α_j are in the same W-orbit.*

Proof: Equivalence of 1 and 2 is proved as in [B, Ch. IV, §1, prop. 3] (Take $S = \{g_1, \ldots, g_n\}$ and use m_{ij} instead of the order of $g_i g_j$). Now α_i and α_j are in the same W-orbit if and only if s_{α_i} and s_{α_j} are conjugate which is equivalent to 2 by the same proposition. $\square$

The following structure theorems of Chevalley and Brieskorn are of fundamental importance:

Theorem 3.1 (Chevalley) *The algebra of W-invariant polynomials on V is itself a polynomial algebra, i.e.:*

$$P[V]^W \cong \mathbb{C}[P_1, \ldots, P_n]$$

Here $P_1, \ldots, P_n$ are homogeneous of degree $d_i := m_i + 1$ and satisfy no algebraic relation. The orbit-space $W \backslash V$ is therefore isomorphic to an affine space [Ch].

Definition 3.2 *The polynomial $D \in \mathbb{C}[X_1, \ldots, X_n]$ given by*

$$D(P_1, \ldots, P_n) = \prod_{\alpha > 0} (\alpha^*)^2$$

is called the discriminant *of R. The zero locus $[D = 0]$ is denoted by Δ. The complement of Δ in $\mathbb{C}^n$ is denoted by X.*

The projection

$$P : V \to \mathbb{C}^n, \ P : v \mapsto (P_1(v), \ldots, P_n(v))$$

is a ramified covering of degree $|W|$ with branch locus Δ. The automorphism group of this covering is exactly W.

The $\mathbb{C}^*$-action on the vectorspace V induces a $\mathbb{C}^*$-action on $\mathbb{C}^n$:

Definition 3.3 *Let* $z = \gcd(d_1, d_2, \ldots, d_n)$ *be the order of the center of* W, *(i.e.* $z \in \{1, 2\}$*). Define a* $\mathbb{C}^*$*-action by:*

$$y \cdot (x_1, x_2, \ldots, x_n) = (y^{d_1}x_1, y^{d_2}x_2, \ldots, y^{d_n}x_n), \textit{ any } y \in \mathbb{C}^*$$

The quotient $\mathbb{C}^* \backslash (\mathbb{C}^n \backslash \{0\})$ *will be denoted by* $\mathbb{P}_d(\mathbb{C}^n)$. *It is called a* weighted projective space *with* weights d_j/z.

Note that $P(y \cdot v) = y \cdot P(v)$ for any $y \in \mathbb{C}^*$ and $v \in V$, so the action restricts to an action on X.

Definition 3.4 *Let* $I \subseteq \{1, 2, \ldots, n\}$ *have* m *elements. The subset*

$$\{P(v) \mid v \in V,\ (\alpha_i, v) = 0 \textit{ iff } i \in I\}$$

of Δ *is called a* $(n-m)$*-dimensional* facet *and if* $I = \{i\}$ *a* type i reflection plane. *The union of all* $(n-1)$*-dimensional facets is called the set of* subregular points.

Theorem 3.2 (Brieskorn) *Pick a basepoint* $x_o \in E^+$, *and write* $y_o = P(x_o)$. *The fundamental group* $\pi_1(X, y_o)$ *is isomorphic to* $B(M)$. *Moreover, if we define loops* G_j *by*

$$G_j : [0,1] \to X,\ G_j(t) = P(x_o + \frac{e^{\pi i t} - 1}{2}(x_o, \alpha_j)\alpha_j)$$

then the homotopy classes of these loops generate the fundamental group and the map $G_j \mapsto g_j$ *extends to an isomorphism.*

Remark 3.1 *If* $Y : [0,1] \to X$ *is given by*

$$Y(t) = P(e^{\frac{2\pi i t}{z}} x_o)$$

then Y *is homotopic to* $(G_1 G_2 \cdots G_n)^{h/z}$. *In particular the latter element is central in the fundamental group. Moreover it even generates the center [D1].*

We will now introduce marked Coxeter diagrams and truncated braid groups.

Definition 3.5 *A marked Coxeter diagram is a Coxeter diagram* M *as before together with* n *integers* $p_1, \ldots, p_n$ *all at least* 2 *such that* $p_i = p_j$ *if the vertices* v_i *and* v_j *are connected in the mod 2 reduced diagram* M. *A marked diagram is graphically denoted by attaching the number* p_i *to the vertex* v_i *if* $p_i > 2$.

From now on, we will write $(M, p_1, \ldots, p_n)$ or simply (M, p), when referring to a marked diagram.

Definition 3.6 *The truncated braid group $B(M,p)$ associated to a marked diagram (M,p) is a group given by generators and relations as follows:*

$$B(M,p) = \langle g_1, \ldots, g_n | (g_i, g_j)^{m_{ij}} = (g_j, g_i)^{m_{ji}},\ g_i^{p_i} = e,\ 1 \leq i \leq j \leq n \rangle$$

We now construct a holomorphic family of representations of the braid group $B(M)$, the so-called *reflection representation*. Throughout these notes, ζ_m denotes the primitive root of unity $\exp(2\pi i/m)$.

Definition 3.7 *A* multiplicity parameter $k : R \to \mathbb{C}$ *is a map which is constant on W-orbits in R. We denote the space of all multiplicity functions by K. For $k \in K$ we will sometimes write k_i instead of k_{α_i}.*

As a $\mathbb{C}$-vector space, K is isomorphic to $\mathbb{C}^t$ if t is the number of W-orbits in R (i.e. $t \in \{1, 2\}$).

Definition 3.8 *The* restricted *multiplicity parameters are defined by:*

$$\begin{array}{rcl} K' & = & \{k \in K \mid 0 < Re(k_i) < \frac{1}{2},\ \textit{for all } i\} \cup \\ & & \{k \in K \mid -\frac{1}{4} < Re(k_i) < \frac{1}{4},\ \textit{for all } i\} \end{array}$$

Then for all $k \in K'$ and i, j such that $m_{ij} > 2$:

$$Re(2(\cos \pi(k_i - k_j) + \cos \frac{2\pi}{m_{ij}})) > 0$$

We define holomorphic functions on K' by:

$$q_j = \exp(-2\pi i k_j),\ \textit{for all } j$$

$$h_{ij} = \begin{cases} q_j^{1/2} + q_j^{-1/2} & \textit{If } i = j \\ 0 & \textit{If } i \neq j \textit{ and } m_{ij} = 2 \\ -(2(\cos \pi(k_i - k_j) + \cos \frac{2\pi}{m_{ij}}))^{1/2} & \textit{If } i \neq j \textit{ and } m_{ij} > 2 \end{cases}$$

Here we take $1^{1/2} = 1$.

Observe that $h_{ij}(k) \neq 0$ if $m_{ij} > 2$ for all $k \in K'$. We will denote the canonical basis of $\mathbb{C}^m$ by $e_1, \ldots, e_m$.

Definition 3.9 *Let for all i, the matrix $r_i \in Mat(n, \mathcal{O}(K'))$ be given by:*

$$(r_i)_{mj} = \delta_{mj} - \delta_{mi} q_i^{1/2} h_{ij}$$

Theorem 3.3 *The map $\varrho : \{g_1, \ldots, g_n\} \to Mat(n, \mathcal{O}(K'))$ mapping g_i to r_i extends to a anti-homomorphism on $B(M)$, i.e. a map ϱ such that $\varrho(g_1 g_2) = \varrho(g_2)\varrho(g_1)$. Moreover, if $k \in K'$ is* real-valued, *then the matrix $H = (h_{ij})$ is real-valued at k, symmetric and satisfies:*

$$\left[{}^t\varrho(g) H \overline{\varrho(g)}\right](k) = H(k), \text{ all } g \in B(M)$$

Proof: As in [CIK, 9.1 & 9.3] if one takes $B(\alpha_r, \alpha_s) = h_{rs}$ and $u_r = q_r^{-1}$. □

Note that for any $k \in K'$ the specialisation $\varrho(k)$ is a *right* representation on $\mathbb{C}^n$. The matrix $r_i(k)$ is a complex reflection with special eigenvalue $-q_i(k)$. If $k \in K'$ is real-valued then $r_i(k)$ is unitary with respect to $H(k)$. Note that if we set $k_i = 0$ for all i, we just get the geometric *right* representation of a Coxeter group (w.r.t. a basis of simple roots), in particular $H(0) = ((\alpha_i, \alpha_j))$.

Remark 3.2 *Suppose we are given two complex reflections a_1, a_2 in $\mathbb{C}^2$:*

$$a_i(e_j) = e_j + s_{ij} e_i$$

If these reflections satisfy a braid relation of m factors then one can prove that

$$s_{12} s_{21} = q_1 + q_2 + (\zeta + \zeta^{-1}) q_1^{1/2} q_2^{1/2}$$

where $q_i = -(1 + s_{ii})$ and ζ is a m^{th} root of unity. Now suppose our Coxeter diagram M is a tree. Then the homomorphism ϱ is up to conjugation the unique one such that

1. *For any $k \in K'$, and any $1 \le i \le n$, the specialisation $\varrho(k)(g_i)$ is a complex reflection.*
2. *The special eigenvectors of $\varrho(k)(g_i)$, $i = 1, \ldots, n$ span $\mathbb{C}^n$.*
3. *The specialisation $\varrho(0)$ is the real reflection representation.*

One can prove this by induction on n. Consider an extremal node from the diagram. This extremal node is connected to exactly one other node of the diagram. This reduces the proof to a rank two situation and there one uses the fact that $s_{12} s_{21} \neq 0$.

Definition 3.10 *If (M, p) is a marked diagram, and we take $k \in K'$ such that $k_i = 1/2 - 1/p_i$ for all i we define the matrix group $G(M, p)$ by*

$$G(M, p) = \langle r_i(k) \mid 1 \le i \le n \rangle$$

The map $g_i \mapsto r_i(k)$ extends to a homomorphism on $B(M, p)$. We call $G(M, p)$ the geometric realisation *of $B(M, p)$.*

We now suppose that M is connected and consists of at least two vertices.

Theorem 3.4 *Define $\underline{k} = (k_1 + \ldots + k_n)/n$ and $\underline{q} = exp(-2\pi i \underline{k})$. Denote the Coxeter element $\varrho(g_1 g_2 \cdots g_n)$ by c_q. The characteristic polynomial of $c_q(k)$ is given by:*

$$P_{c_q(k)}(T) = \prod_{j=1}^{n} (T - \underline{q}\zeta_h^{m_j})$$

Proof: By remark 3.1 we know that $c_q(k)^h$ commutes with $r_1(k), \ldots, r_n(k)$. This implies that it is diagonal w.r.t. the basis $e_1, \ldots, e_n$. If $m_{ij} > 2$ then a computation shows that the diagonal entries on the places i and j must be equal. This implies that $c_q(k)^h$ is in fact a scalar times the identity because M is connected. Say $c_q(k)^h = \mu \cdot 1_n$, then by taking determinants we see $\underline{q}^{nh} = \mu^n$, so $\mu = \zeta_n^m \underline{q}^h$ for some m. Setting $k_i = 0$ for all i, shows in fact that $\mu = \underline{q}^h$. So all eigenvalues of $c_q(k)$ are of the form $\zeta_h^m \underline{q}$. Again considering $k_i = 0$ finally proves the theorem. □

Corollary 3.1 *If $k \in K'$ then $\varrho(k)$ is a* reducible *representation of $B(M)$ iff $\underline{q} = \zeta_h^{m_j}$ for some j. Moreover, if it is reducible then the only non trivial invariant subspace of $\mathbb{C}^n$ is one dimensional.*

Proof: Because $h_{ij} \neq 0$ if $m_{ij} > 2$ a non trivial invariant subspace of $\mathbb{C}^n$ must be kept pointwise fixed by generators $r_j(k)$. In particular the Coxeter element $c_q(k)$ must have an eigenvalue 1. This is the case iff $\underline{q} = \zeta_h^{m_j}$ for some j. On the other hand, any eigenvector of $c_q(k)$ with eigenvalue 1 is kept fixed by all reflections $r_j(k)$. This proves the corollary. □

We now consider Coxeter elements associated with subdiagrams of M. Let I be some subset of $\{1, \ldots, n\}$ such that the subdiagram M' of M spanned by the vertices v_i, $i \in I$ is connected. If $\#I = m$, $I = \{i_1, \ldots, i_m\}$ then let $c_q^I = \varrho(g_{i_1} \cdots g_{i_m})$

Theorem 3.5 *If $k \in K'$ is such that $\varrho(k)$ is irreducible and $c_q^I(k)$ has a non zero fixed point in the subspace*

$$C^I := Span_{\mathbb{C}}\{e_{i_1}, \ldots, e_{i_m}\} \subseteq \mathbb{C}^n$$

then $c_q^I(k)$ is not semisimple. If $c_q^I(k) = S + N$ is its Jordan decomposition in a semisimple and nilpotent part respectively, then rank$(N) = 1$.

Proof: Because $\varrho(k)$ is irreducible the fixed point set of the endomorphism $c_q^I(k)$ is a linear subspace of dimension $n-m$ (indeed $c_q(k)$ has no non-zero fixed point). But by our assumption the fixed point set intersects C^I non trivially (and hence in a one dimensional subspace by theorem 4). Now $c_q^I(k)$ restricted to the $n-1$ dimensional space

$$C^I + \text{Fixed points}$$

is semisimple. Clearly $1-c_q^I(k)$ maps $\mathbb{C}^n$ into C^I so $c_q^I(k)$ itself is not semisimple. This proves the theorem. $\square$

For real valued $k \in K'$ we now compute the signature of the invariant Hermitean form.

Theorem 3.6 *Let the matrix H be defined as above. The determinant of H is given by:*

$$det(H) = 2^n \prod_{i=1}^{n} (\cos \pi \underline{k} + \cos \frac{m_i \pi}{h})$$

Proof: Due to Coxeter [C2]. From an exercise in Bourbaki ([B], Ch. V, §6, exerc. 3,4) we know that:

$$det(H) = \underline{q}^{-n/2} det(1 - c_q)$$

Using theorem 4 we obtain:

$$det(H) = \prod_{j=1}^{n} (\underline{q}^{-1/2} - \underline{q}^{1/2} \zeta_h^{m_j}) =$$

$$= \prod_{j=1}^{n} (\underline{q}^{-1/4} - \underline{q}^{1/4} \zeta_{2h}^{m_j})(\underline{q}^{-1/4} + \underline{q}^{1/4} \zeta_{2h}^{m_j}) =$$

$$= \prod_{j=1}^{n} (\underline{q}^{-1/4} + \underline{q}^{1/4} \zeta_{2h}^{-m_j})(\underline{q}^{-1/4} + \underline{q}^{1/4} \zeta_{2h}^{m_j}) =$$

$$= \prod_{j=1}^{n} (\underline{q}^{-1/2} + \underline{q}^{1/2} + \zeta_{2h}^{m_j} + \zeta_{2h}^{-m_j}) = 2^n \prod_{j=1}^{n} (\cos \pi \underline{k} + cos \frac{m_j \pi}{h})$$

Here we used the fact that $m_j + m_{n+1-j} = h$. $\square$

Corollary 3.2 *If $k \in K$ is real valued and $0 \leq k_j < 1/2$ for all j then $H(k)$ is*

1. *positive definite iff $0 < 1 - h\underline{k} \leq 1$.*
2. *parabolic (i.e. positive semi definite with one-dimensional kernel) iff $1 - h\underline{k} = 0$.*
3. *hyperbolic (i.e. has signature $(n-1, 1)$) iff $1 - m_2 < 1 - h\underline{k} < 0$.*

Proof: Again from the same exercises in [B] one can deduce that, in case $k_i = k$ all i, the eigenvalues of H are exactly $2(\cos \pi \underline{k} - \cos \frac{m_j \pi}{h})$. So in this case the signature of H can be read off as indicated. However, we know that the determinant of H does only depend on $\underline{k}$. Hence the signature of H does not change if we vary k_i in such a way that $\underline{k}$ remains constant. This proves the corollary. □

Definition 3.11 *If (M,p) is a connected marked diagram and $k \in K'$ is such that $k_i = 1/2 - 1/p_i$, we call*

$$\nu(k) = 1 - h\underline{k}$$

the exponent *of the marked diagram.*

By using the well known property ([B]) that the cyclic group generated by the Coxeter element c of W has n orbits of length h on the roots R one deduces:

$$\nu(k) = 1 - \frac{1}{n} \sum_{\alpha \in R} k_\alpha$$

Definition 3.12 *We denote the transpose of ϱ by ϱ^*, i.e.:*

$$\varrho^*(g) = {}^t\varrho(g), \ \ g \in B(M)$$

In particular for any $k \in K$, $\varrho^*(k)$ is a (left) representation of $B(M)$. Let $H^* \in Mat(n, \mathcal{O}(K'))$ be given by:

$$H^* = \det(H) H^{-1}$$

(This is well defined, moreover this is just the minor matrix of H).

Theorem 3.7 *If $k \in K'$ is realvalued, then $H^*(k)$ is a non-trivial invariant Hermitian form for the transpose $\varrho^*(k)$ at k. Moreover, if $H(k)$ is positive definite, then $H^*(k)$ is also positive definite. If $H(k)$ is parabolic, then $H^*(k)$ is positive semi-definite, and has an $n-1$ dimensional kernel. If $H(k)$ is hyperbolic, then te signature of $H^*(k)$ is $(1, n-1)$.*

Proof: Because $H(k)$ is at least of rank $n-1$, the matrix $H^*(k)$ is at least of rank one. Then $H^*(k)$ is clearly a non-trivial Hermitian form for $\varrho^*(k)$. The statements for the elliptic and hyperbolic cases are clear. If $H(k)$ is parabolic, the statement follows from the equality $H(k)H^*(k) = 0$. □

To end this section we construct the *logarithmic reflection representation* of $B(M)$. Let $k \in K'$ be such that $\nu(k) = 0$. Then $\varrho(k)$ has a non-zero fixed point in $\mathbb{C}^n$ unique upto scalar multiples. Let $\beta_j = \varrho(k, g_{j+1} \cdots g_n)e_j$ for $j \in$

$\{1,\ldots,n\}$. Let $x_j \in \mathbb{C}$ be such that $\lambda := \sum_j x_j \beta_j$ is a non-zero fixed point of $\varrho(k)$. Define endomorphisms $\tilde{r}_1(k),\ldots,\tilde{r}_n(k)$ of $\mathbb{C}^{n+1}$ by

$$\tilde{r}_i(k)e_j = \begin{cases} r_i(k)e_j & \text{If } j \leq n \\ e_{n+1} + x_i e_i & \text{If } j = n+1 \end{cases}$$

Then the map $\tilde{\varrho}(k) : B(M) \to \mathrm{End}(\mathbb{C}^{n+1})$, $\tilde{\varrho}(k, g_j) := \tilde{r}_j(k)$ extends to a right representation of $B(M)$ called the *logarithmic reflection representation.* One checks that $\tilde{\varrho}(k, g_1 \cdots g_n)e_{n+1} = e_{n+1} + \lambda$ and hence $\tilde{\varrho}(k, g_1 \cdots g_n)$ has a nilpotent part of rank one.

Lemma 3.2 *The only non trivial invariant subspaces of the logarithmic reflection representation are $\mathbb{C}\lambda$ and $Span_{\mathbb{C}}\{e_1,\ldots,e_n\}$. Here λ denotes a fixed vector (unique upto a scalar).*

Proof: The logarithmic representation restricted to $A := \mathrm{Span}_{\mathbb{C}}\{e_1,\ldots,e_n\}$ is equivalent to the reflection representation. Hence the only invariant subspaces contained in A are $\{0\}$, $\mathbb{C}\lambda$ and A. If B is a non-trivial invariant subspace not contained in A, then $B \cap A$ is at most one dimensional. hence B is at most two dimensional and contains a vector of the form $e_{n+1} + a$, $a \in A$. The endomorphism $1 - \tilde{\varrho}(k, g_j)$ maps B into $B \cap \mathbb{C}e_j = \{0\}$. Hence B must be kept pointwise fixed by the logarithmic representation. However, let the central element act on $e_{n+1} + a \in B$ to obtain

$$\tilde{\varrho}(k, (g_1 \cdots g_n)^h)(e_{n+1} + a) = e_{n+1} + a + x\lambda$$

for some non zero $x \in \mathbb{C}$. This shows that every non trivial invariant subspace is contained in A. $\square$

3.3 The Dunkl connection

Notations as in the previous section. We will assume that the root system R is *irreducible* and of full rank in E. Let $(k_\alpha \mid \alpha \in R)$ be a W-invariant multiplicity parameter on the roots. Let $\tau : W \to \mathrm{End}(H)$ be a representation of the Coxeter group W. Denote the sheaf of local holomorphic sections in the trivial bundle $V^{reg} \times H$ over V^{reg} by $\mathcal{A}^0(H)$. Let

$$\mathcal{A}^1(H) = \Omega^1(V^{reg}) \otimes_{\mathcal{O}_{V^{reg}}} \mathcal{A}^0(H)$$

The *Dunkl connection* on $\mathcal{A}^0(H)$ is given by:

$$\nabla(k) : \mathcal{A}^0(H) \to \mathcal{A}^1(H)$$

$$\nabla(k)h = \sum_{\alpha>0} \frac{k_\alpha}{\alpha^*} d\alpha^* \otimes (1 - \tau(s_\alpha))h$$

Note that by describing how $\nabla(k)$ acts on the constant sections it is completely determined as a connection. The action of W on V^{reg} naturally extends to an action on $\mathcal{A}^{\cdot}(H)$ by acting as τ on the constant global sections.

Theorem 3.8 (Dunkl) *The connection $\nabla(k)$ commutes with the W-action and has zero curvature, i.e. is completely integrable.*

Proof: Omitted.□

We will concentrate on the case that τ is the reflection representation of W. For technical reasons which will become clear in a moment we take the reflection representation on the differentials $\Lambda^1 V$ rather than on V itself. In particular $\mathcal{A}^{\cdot}(\Lambda^1 V) = \Omega^{\cdot} \otimes \Omega^1$. The reflection representation acts by

$$\tau(w)d\lambda^* = d(w\lambda)^*$$

for all $w \in W$, $\lambda \in V$. Substituting this in the formula for the Dunkl connection yields:

$$\nabla(k)d\lambda^* = \sum_{\alpha>0} \frac{k_\alpha(\alpha,\lambda)}{\alpha^*} d\alpha^* \otimes d\alpha^*$$

Let $\xi_1, \ldots, \xi_n \in E$ denote an orthonormal basis for V. One checks that a local section $\omega = \sum_i f_i d\xi_i^*$ is *flat* for the Dunkl connection iff

$$df_i + \sum_{\alpha>0} \frac{k_\alpha \omega(\partial_\alpha)(\alpha,\xi_i)}{\alpha^*} d\alpha^* = 0$$

for all i. To obtain results about flat sections we need the following lemma.

Lemma 3.3 *For any $\lambda, \eta \in V$ the following equality holds:*

$$\sum_{\alpha>0} k_\alpha(\alpha,\lambda)(\alpha,\eta) = \delta \cdot (\lambda,\eta)$$

Here $\delta = \delta(k)$ is given by:

$$\delta(k) = \frac{2}{n} \sum_{\alpha>0} k_\alpha$$

Proof: The sum on the left hand side is a W-invariant bilinear symmetric form on V. Because W acts irreducible on V it must be a constant δ times the form $(\cdot,\cdot)$. And we deduce:

$$\delta n = \sum_{i=1}^{n} \delta(\xi_i,\xi_i) = \sum_{\alpha>0} k_\alpha(\sum_{i=1}^{n}(\alpha,\xi_i)\xi_i,\alpha) =$$
$$= \sum_{\alpha>0} k_\alpha(\alpha,\alpha) = 2\sum_{\alpha>0} k_\alpha$$

This proves the lemma. □

Corollary 3.3 *If $\omega = \sum_i f_i d\xi_i^*$ is a flat local section then*

$$\sum_{i=1}^{n} \xi_i^* df_i = -\delta\omega$$

Proof: Using flatness of ω we get

$$\sum_{i=1}^{n} \xi_i^* df_i = -\sum_{\alpha>0} k_\alpha \omega(\partial_\alpha) d\alpha^* =$$

$$= -\sum_{i,j} f_i \sum_{\alpha>0} k_\alpha (\alpha, \xi_i)(\alpha, \xi_j) d\xi_j^* = -\delta \sum_i f_i d\xi_i^* = -\delta\omega$$

This proves the corollary. □

Denote the Euler field $\sum_i \xi_i^* \partial_{\xi_i}$ on V^{reg} by $\mathcal{E}$.

Theorem 3.9 *Let ω be a flat local section and $\nu = \nu(k) = 1 - \delta(k)$. Then $d[\omega(\mathcal{E})] = \nu\omega$ and $\mathcal{E}\omega(\mathcal{E}) = \nu\omega(\mathcal{E})$.*

Proof: Let f_i be such that $\omega = \sum_i f_i d\xi_i^*$. Then

$$d[\omega(\mathcal{E})] = d[\sum_i f_i \xi_i^*] = \omega + \sum_i \xi_i^* df_i = \nu\omega$$

and

$$\mathcal{E}\omega(\mathcal{E}) = \mathcal{E} \sum_i f_i \xi_i^* = \omega(\mathcal{E}) + \sum_{i,j} \partial_{\xi_j} f_i \xi_j^* \xi_i^* =$$

$$= \omega(\mathcal{E}) + \sum_i \xi_i^* df_i(\mathcal{E}) = \nu\omega(\mathcal{E})$$

This proves the theorem. □

Note that the second statement of the theorem just states that the holomorphic function $\omega(\mathcal{E})$ is homogeneous of degree ν.

Theorem 3.10 *The $\mathbb{C}$-linear operator $\nabla(k)d$ on $\mathcal{O}_{V^{reg}}$ has locally an $(n+1)$-dimensional kernel everywhere on V^{reg}.*

Proof: First note that f is in the kernel of $\nabla(k)d$ iff

$$\left[\partial_\lambda \partial_\eta + \sum_{\alpha>0} \frac{k_\alpha}{\alpha^*} (\alpha, \lambda)(\alpha, \eta) \partial_\alpha\right] f = 0$$

for all $\lambda, \eta \in V$. We will call such an f a *solution* of $\nabla(k)d$. Such a solution is completely determined by its first order Taylor part. Hence the kernel is at most $(n+1)$-dimensional.

Now assume that the multiplicity parameter k is such that $\nu = \nu(k) \neq 0$. Then if ω locally runs over the flat sections of $\nabla(k)$, the functions $\omega(\mathcal{E})$ span an n-dimensional subspace of the kernel of $\nabla(k)d$ all of homogeneous degree $\nu \neq 0$. Together with the constants, this yields that the kernel is exactly $(n+1)$-dimensional. The coefficients of solutions depend polynomially on k, so the operator $\nabla(k)d$ has an $(n+1)$-dimensional kernel for all values of k. $\square$

There is a nice way to reformulate this result in terms of connections. Consider the following mapping (sheafs are over V^{reg}):

$$\tilde{\nabla}(k) : \mathcal{O} \oplus \Omega^1 \to \Omega^1 \otimes (\mathcal{O} \oplus \Omega^1)$$

$$\tilde{\nabla}(k)(f + \omega) = (df - \omega) \otimes 1 + \nabla(k)\omega$$

One readily checks that $\tilde{\nabla}(k)$ is a connection.

Theorem 3.11 *The connection $\tilde{\nabla}(k)$ is completely integrable and regular singular along the reflection planes.*

Proof: A local section $f + \omega$ is flat iff $\omega = df$ and $\nabla(k)df = 0$. By the previous theorem, there are sufficiently many of such f to conclude complete integrability. That the connection is regular singular is clear from the explicit formula for $\nabla(k)$.$\square$

This result shows that the theory of regular singular integrable connections applies to solutions of $\nabla(k)d$.

Remark 3.3 *One checks that a solution f of $\nabla(k)d$ also satisfies*

$$\left[\sum_{i=1}^{n} \partial_{\xi_i}^2 + \sum_{\alpha>0} \frac{2k_\alpha}{\alpha^*}\partial_\alpha\right] f = 0$$

The operator between square brackets is a deformation in the parameter k of the euclidean Laplace operator and is sometimes denoted by $L(k)$. If R is a crystallographic root system and k takes some specific values, $L(k)$ turns up as the radial part of the laplacian on the tangent space of a Riemannian symmetric space. The operator $L(k)$ (R defined over $\mathbb{R}$, k arbitrary) was studied extensively by E. Opdam in a paper about multivariable Bessel functions associated to root systems [O].

Observe that the group W acts naturally on $\mathcal{A}^{\cdot}(\mathbb{C} \oplus \Lambda^1 V)$ and this action commutes with $\tilde{\nabla}(k)$. This enables us to construct the monodromy representation for the *quotient* $W \backslash V^{reg}$ by analytic continuation of solutions of $\nabla(k)d$.

Take $v \in V^{reg}, k \in K$. Consider ∇d as an operator on the stalk of holomorphic germs $\mathcal{O}_{(k,v)}$ (i.e. view the parameter k in ∇ as an additional variable). It is well known that the solutions then form a free $\mathcal{O}_k$ module of rank $n+1$.

Hence local solutions of ∇d near v can be considered as a vector bundle $\mathcal{F}_v$ over K. Any $w \in W$ induces a canonical vector bundle isomorphism ϕ^w of $\mathcal{F}_v$ onto $\mathcal{F}_{w(v)}$. If S is a regular W-orbit we can identify the bundles $\mathcal{F}_v$, $v \in S$ by the isomorphisms ϕ^w. This yields a vector bundle $\mathcal{F}_S$ over K of rank $n+1$. The fibre of $\mathcal{F}_S$ at $k \in K$ will be denoted by $\mathcal{F}_S(k)$. Lifting loops in $W\backslash V^{reg}$ to W by the projection together with analytic continuation yields a canonical anti-homomorphism

$$\rho : \pi_1(W\backslash V^{reg}, S) \cong B(M) \to \mathrm{End}(\mathcal{F}_S).$$

By specialising k we get a *right* representation $\rho(k)$ on the vector space $\mathcal{F}_S(k)$. We write $\rho(k,g)$ for $\rho(k)(g)$. To study these representations we will compute the exponents of $\nabla(k)d$ along the reflection planes.

Lemma 3.4 *Suppose $k \in K'$. Along a plane $\alpha^* = 0$, the exponents of $\nabla(k)d$ are 0 with multiplicity n and $1 - 2k_\alpha$ with multiplicity one.*

Proof: That these are the only two exponents occuring along $\alpha^* = 0$ follows by letting $L(k)$ act on a solution of the form $(\alpha^*)^\epsilon f$ for some exponent ϵ and a holomorphic function f. If we take $k = 0$ then solutions are just polynomials of degree at most one, i.e. exponents 0 and 1 appear with multiplicity n and 1 respectively. Because the exponents 0 and $1 - 2k_\alpha$ do not coincide if k ranges over K', these multiplicities can not change. $\square$

Theorem 3.12 *Let $k \in K'$ and $1 - m_2 < Re(\nu(k)) \leq 1$. If $\nu(k) \neq 0$ then $\rho(k)$ is equivalent to the sum of the trivial representation and the reflection representation $\varrho(k)$ of $B(M)$ (as right representations). If $\nu(k) = 0$ then $\rho(k)$ is equivalent to the logarithmic reflection representation.*

Proof: If $\nu(k) \neq 0$ then the representation $\rho(k)$ splits in the trivial for the constant function and an n-dimensional for the homogeneous degree ν part. However, $\rho(k, g_j)$ is a complex reflection with eigenvalue $-q_j$, and for $k = 0$, $\rho(0)$ splits as indicated. The (continuous) deformation in k can only be done in one way as we observed in the previous section. This settles the $\nu \neq 0$ case.

If $\nu(k) = 0$ then for any $\nabla(k)$-flat section ω we computed that $d[\omega(\mathcal{E})] = 0$. Hence $\omega(\mathcal{E})$ is a constant for all such sections. At any point in V^{reg} the value of a $\nabla(k)$ flat section can be prescribed freely, showing that $\omega(\mathcal{E})$ is not zero for all flat sections. The exterior derivative d maps the solutions of $\nabla(k)d$ onto the $\nabla(k)$ flat sections. A solution f is homogeneous of degree 0 iff $(df)(\mathcal{E})$ vanishes at some point in V^{reg} (because it is then constant and equal to zero). However, at any point in V^{reg} the first order part of f can be prescribed freely. This implies that the solutions of $\nabla(k)d$ of homogeneous degree $\nu(k)$ form an n-dimensional subspace for all values of k. The vector space of germs of homogeneous solutions of $\nabla(k)d$ at $v \in V^{reg}$ is denoted by $\mathcal{E}_v(k)$.

The operator $\mathcal{E} - \nu(k)$ is an endomorphism of $\mathcal{F}_S$ and its kernel is a subbundle of $\mathcal{F}_S$ of rank n, invariant under monodromy. Denote this bundle by $\mathcal{E}_S$. As endomorphism of $\mathcal{E}_S$, the element $\rho(g_1 \cdots g_n)$ has the characteristic polynomial:

$$\prod_{j=1}^{n} (T - \underline{q}\zeta_h^{m_j})$$

Specializing k in this polynomial at some fixed value, always yields a polynomial with n distinct roots. Note that bundles over K are trivial and hence there exists a global non vanishing section $f_c \in \Gamma(\mathcal{F}_S)$ such that

$$\rho(g_1 \cdots g_n) f_c = \exp(2\pi i \frac{\nu(k)}{h}) f_c$$

The vector $f_c(k)$ being unique up to a scalar multiple, we may assume that $f_c(k) = 1$ (as a constant function) if $\nu(k) = 0$. Similarly we get non vanishing sections $e_1, \ldots, e_n$ on K' in $\mathcal{E}_S$ such that

$$\rho(g_i) e_j = \sum_{l=1}^{n} (r_i)_{lj} e_l$$

for all i, j. Consider the function

$$\ell = \frac{f_c - 1}{\nu(k)}$$

Note that it is in $\mathcal{O}_{K \times \{v\}}$, because $f_c - 1$ vanishes identically if $\nu(k) = 0$. By continuity in k, we conclude $\nabla d\ell = 0$, so ℓ is in fact a global section in the bundle $\mathcal{F}_S$. Analytic continuation gives:

$$\rho(g_1 \cdots g_n)\ell = \frac{\exp(2\pi i \nu(k)/h) f_c - 1}{\nu(k)} = \exp(2\pi i \frac{\nu(k)}{h})\ell + \frac{\exp(2\pi i \nu(k)/h) - 1}{\nu(k)}$$

For $\nu(k) = 0$ we get

$$\rho(k, g_1 \cdots g_n)\ell(k) = \ell(k) + \frac{2\pi i}{h}$$

Similarly one shows that (continuing $\ell(k)$ through V^{reg})

$$\ell(k, x\lambda) = \ell(k, \lambda) + \log(x)$$

for all $x \in \mathbb{C}^*$, $\lambda \in V^{reg}$. All transformations $\rho(g_j)$ are complex reflections and the action of $\rho(g_1 \cdots g_n)$ on ℓ shows in particular that if $\nu(k) = 0$

$$1 \in \mathrm{Span}_{\mathbb{C}}\{e_1(k), \ldots, e_n(k)\}$$

This implies that the functions $e_j(k)$ are linearly independent (over $\mathbb{C}$) because up to a scalar there is exactly one linear combination of these functions which is monodromy invariant. This shows that $\rho(k)$ is the logarithmic reflection representation if $\nu(k) = 0$. $\square$

3.4 The evaluation mapping

Let S be a regular W-orbit and U a simply connected neighborhood of S in $W\backslash V^{reg}$. By identifying the dual bundles $\mathcal{F}_v^*$, $v \in S$ by the duals of the isomorphisms ϕ^w we get the dual bundle $\mathcal{F}_S^*$. We identify $\pi_1(W\backslash V^{reg}, S)$ and $B(M)$ using Brieskorn's theorem and sometimes call elements of $B(M)$ loops. Transposing ρ yields a (left) representation

$$\rho^* : B(M) \to \mathrm{End}(\mathcal{F}_S^*).$$

There is a canonical holomorphic mapping $ev : K \times U \to \mathcal{F}_S^*$ into the dual bundle given by:

1. For all $u \in U$, $k \mapsto ev(k,u)$ is a global section in $\mathcal{F}_S^*$.
2. $ev(k,u)(f) := f(u)$. Here f is an element of the fibre $\mathcal{F}_S(k)$.

Note that the evaluation $f(u)$ in 2 is well defined and indeed defines a section in $\mathcal{F}_S^*$.

The name ev stands for *evaluation*. This evaluation mapping extends to a multi valued holomorphic mapping ev of $K \times (W\backslash V^{reg})$ into $\mathcal{F}_S^*$. For fixed $k \in K$ we denote by $ev(k)$ the multi valued holomorphic mapping $ev(k,\cdot)$ of $W\backslash V^{reg}$ into the dual of the fibre $\mathcal{F}_S(k)$.

Before stating some properties of the evaluation mapping we introduce the *Wronskian* of $\nabla(k)d$. Let $\lambda_1, \ldots, \lambda_n \in V$ be a basis and let $f_0, \ldots, f_n$ be a basis of local solutions of $\nabla(k)d$.

Definition 3.13 *The* Wronskian *of $\nabla(k)d$ is defined up to non-zero scalar multiplication by:*

$$J := \det \begin{pmatrix} f_0 & \partial_{\lambda_1} f_0 & \ldots & \partial_{\lambda_n} f_0 \\ f_1 & \partial_{\lambda_1} f_1 & \ldots & \partial_{\lambda_n} f_1 \\ \vdots & \vdots & & \vdots \\ f_n & \partial_{\lambda_1} f_n & \ldots & \partial_{\lambda_n} f_n \end{pmatrix}$$

Note that J is indeed independent of the choice of basis up to a non zero scalar multiple.

Lemma 3.5 *The Wronskian of $\nabla(k)d$ is given by:*

$$J = \prod_{\alpha > 0} (\alpha^*)^{-2k_\alpha}$$

Proof: From the definition of the Wronskian as a determinant one deduces that J satisfies
$$\left[\partial_\xi + \sum_{\alpha>0} \frac{2k_\alpha(\alpha,\xi)}{\alpha^*}\right] J = 0$$
for all $\xi \in V$. The proposed product formula for J satisfies all these equations. This proves the lemma. $\square$

By identifying $W\backslash V^{reg}$ and X using the Chevalley projection P we will henceforth consider ev as a multivalued holomorphic mapping on $K \times X$.

Theorem 3.13 *For any $k \in K$ the mapping $\mathrm{ev}(k)$ satisfies the following properties:*

1. *It maps locally biholomorphically into an affine subspace $A(k)$ of $\mathcal{F}_S^*(k)$.*
2. *Continuing $\mathrm{ev}(k)$ along a loop $g \in B(M)$ yields $\rho^*(k,g)\mathrm{ev}(k)$.*
3. *Near a subregular point x, we can pick local coordinates $y_1, \dots, y_n$ and certain linear coordinates of $\mathcal{F}_S^*(k)$ such that near x, the evaluation mapping has the following form:*
$$\mathrm{ev}(k) = (y_1^{\frac{1}{2}-k_j}, y_2, \dots, y_n, 1)$$

Proof: Evaluation of the constant function 1 at any point yields 1, proving that it maps into an affine subspace of $\mathcal{F}_S^*(k)$ which we will denote by $A(k)$. That evaluation $ev(k)$ is locally biholomorphic everywhere follows from the fact that df for a solution f of $\nabla(k)d$ can be prescribed freely at any point of V^{reg}. This proves 1. Statement 2 is clear.

Near x, there are holomorphic functions
$$x_1, \dots, x_n, 1$$
such that none of them is (locally) divisible by the discriminant D and the pullbacks by P of the following functions form a basis of $\mathcal{F}_{Wy}(k)$ for y near x:
$$D^{\frac{1}{2}-k_j} x_1, x_2, \dots, x_n, 1$$
The Wronskian takes the form (with $P_1, \dots, P_n$ the standard coordinates on $\mathbb{C}^n$):
$$D^{-k_j} x_1 \cdot \det\left(\frac{\partial(D, x_2, \dots, x_n)}{\partial(P_1, \dots, P_n)}\right) + \text{ higher order terms of } D$$
Hence both x_1 and $\det(\frac{\partial(D,x_2,\dots,x_n)}{\partial(P_1,\dots,P_n)})$ are non-vanishing near x. The following are indeed coordinates near x:
$$y_1 = D \cdot x_1^{(\frac{1}{2}-k_j)^{-1}}, y_2 = x_2, \dots, y_n = x_n$$

With respect to these coordinates, the evaluation mapping can be written as stated in 3. □

Corollary 3.4 *Let $y_1, \ldots, y_n$ be coordinates near x as above. Suppose that $k_j = \frac{1}{2} - \frac{1}{p_j}$, for some $p_j \in \{2, 3, \ldots\}$. The composition*

$$ev(k) \circ (y_1^{p_j}, y_2, \ldots, y_n)$$

extends locally biholomorphically to a neighborhood of x. (It is in fact the identity mapping).

Proof: This is clear if we write $ev(k)$ in the coordinates $y_1, \ldots, y_n$ also. □

Our local analysis of the evaluation mapping reveals its branching behaviour at subregular points of the discriminant. We use this analysis later on to study branching behaviour of coverings at the other singular points also.

Consider the subbundle $\mathcal{E}_S$ of $\mathcal{F}_S$ introduced in the previous section. It is stable under monodromy and hence we also have a monodromy representation ρ^* on $\mathcal{E}_S^*$. The natural restriction mapping

$$Res(k) : \mathcal{F}_S^*(k) \to \mathcal{E}_S^*(k)$$

is a surjective intertwining operator. If $\nu(k) \neq 0$ the vector space $\mathcal{E}_S(k)$ is complemented by the constant functions in $\mathcal{F}_S(k)$. In this case, restriction induces an *equivalence* between the annihilator of the constant functions and $\mathcal{E}_S^*(k)$.

In section 3.5 and 3.7 we will study *restricted* evaluation $Rev := Res \circ ev$ instead of evaluation itself because the constant functions do not play an important role there. The constants do play an important role however in the parabolic theory. Hence in section 3.6 we will study the mapping ev.

3.5 The elliptic case

Throughout this section we assume that we have chosen the marks at the nodes of a finite irreducible Coxeter diagram in such a way that it becomes elliptic. This means that the exponent of the marked diagram (and hence of all its connected subdiagrams) is positive, or equivalently, that the invariant Hermitean form H for the standard reflection representation is positive definite. The corresponding multiplicity parameter k is given by $k_i = 1/2 - 1/p_i$.

Let $\pi : \widetilde{X} \to X$ be the universal covering of the discriminant complement. Identify $\mathrm{Aut}(\widetilde{X}|X)$ and $B(M)$. We lift the mapping Rev to a single valued mapping $\widetilde{ev} : \widetilde{X} \to \mathcal{E}_S^*(k)$. Let $\Gamma(p)$ be the smallest normal subgroup of $B(M)$

containing $g_1^{p_1}, \ldots, g_n^{p_n}$. Let $X_u(p) := \Gamma(p)\backslash\widetilde{X}$. Any p_j-fold loop around a type j reflection plane induces the identity automorphism of $X_u(p)$ and it is universal with respect to this property. The projection π induces a projection $\pi_u : X_u(p) \to X$. We refer to $X_u(p)$ as the *universal covering of X of local degree p*. In the elliptic case, this covering can be extended very nicely, in the sense of the following theorem.

Theorem 3.14 *Suppose $k \in K$ is given by $k_j = 1/2 - 1/p_j$ for some integers $p_j \in \mathbb{Z}_{\geq 2}$. If $\nu(k) > 0$ there exists a ramified covering $\pi_r : X_r(p) \to \mathbb{C}^n$, branching along Δ with local degrees p_j, such that $X_u(p) = \pi_r^{-1}(X)$ and π_u is just the restriction of π_r.*

Proof: During the proof we construct the commuting diagram shown in figure 3.1, consisting of covering maps and several functions related to evaluation.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{\mathrm{ev}}"] \arrow[d] & \mathcal{E}_S^*(k) & \\
X_u(p) \arrow[r] \arrow[d] \arrow[ur, "\mathrm{ev}_u"'] & X_r^*(p) \arrow[r] \arrow[d] \arrow[u, "\mathrm{ev}_r^*"'] & X_r(p) \arrow[d] \\
X \arrow[r] & \mathbb{C}^n\backslash\{0\} \arrow[r] & \mathbb{C}^n
\end{tikzcd}$$

Figure 3.1: The elliptic case.

We prove the theorem by induction on the rank n. In rank one this is just the remark that the mapping

$$\pi^p : \mathbb{C}^* \to \mathbb{C}^*, \quad \pi^p : z \mapsto z^p$$

can be extended to $\mathbb{C}$ (with image $\mathbb{C}$). Now assume that such branched coverings exist for all elliptic diagrams of rank less than n. Take a singular point $x \in \Delta\backslash\{0\}$. There exist local coordinates on a neighborhood U of x such that $U \cap X$ is biholomorphically equivalent with a product

$$U \cap X \cong \Delta_1^m \times U_1 \times \ldots \times U_s$$

Here each U_j denotes the complement of a discriminant of a parabolic irreducible sub root system in a polydisc. For example, take the diagram of A_3, number the corresponding simple roots from left to right.

If we take x the Chevalley image of a point stable exactly under the first two simple reflections, then a small neighborhood would look like

$$U \cap X \cong \Delta_1 \times (\Delta_1^2\backslash\Delta(A_2))$$

where $\Delta(A_2)$ denotes a discriminant of type A_2. If x is the Chevalley image of a point stable exactly under the first and third simple reflection such a neighborhood would look like:

$$U \cap X \cong \Delta_1 \times \Delta_1^* \times \Delta_1^*$$

Where Δ_1^* denotes the punctured disc. Because the subdiagrams have lower rank and are of elliptic type, we conclude by induction that there exists a ramified covering

$$\pi_U : X_{ram}(p, U) \to U$$

such that $\pi_U^{-1}(U \cap X)$ is *universal of degree p* over $U \cap X$. While $\widetilde{\mathrm{ev}}$ branches with the right orders along Δ it descends to a locally biholomorphic function ev_u on $\pi_U^{-1}(U \cap X)$. Moreover, considering theorem 3.13, ev_u extends locally biholomorphically over the π_U pre image of all sub regular points in U.

The preimage of the non subregular part of U is stratified in strata which are all of codimension at least two. Using the isomorphism theorem from section 1.1 we conclude that ev_u extends locally biholomorphically over all of $X_{ram}(p, U)$ to a mapping ev_r^*.

Every covering automorphism of $X_{ram}(p, U)$ fixes the pre image of $x \in U$. Hence the only automorphism which fixes the mapping ev_r^* is the identity. Any connected component of the pre image $\pi_u^{-1}(U) \subseteq X_u(p)$ is a quotient of the universal degree p covering $\pi_U^{-1}(U \cap X)$. However, because ev_r^* must be constant on fibres of this quotient mapping, we conclude by the previous remark that a connected component of $\pi_u^{-1}(U)$ is in fact *isomorphic* to this universally branched covering. Hence all local extensions fit together and we get a ramified covering $\pi_r^* : X_r^*(p) \to \mathbb{C}^n \backslash \{0\}$ containing $X_u(p)$ as a subcovering. Moreover, ev_u extends locally biholomorphically over all of $X_r^*(p)$ to a mapping ev_r^*.

It remains to show that we can extend $X_r^*(p)$ over the origin. We prove this by using a topological argument and again Hartog's theorem. It turns out that ev_r^* is *globally* biholomorphic on $X_r^*(p)$ with image $\mathcal{E}_S^*(k)\backslash\{0\}$. Let $e_1, \ldots, e_n$ be a basis of $\mathcal{E}_S(k)$, where $e_1, \ldots, e_n$ are chosen as in the end of section 3.3. Let $e_1^*, \ldots, e_n^*$ be the dual basis of $\mathcal{E}_S^*(k)$. As in definition 3.12 let $H^*(e_i^*, e_j^*) = H_{ij}^*$ be a $\rho^*(k)$-invariant hermitian form. We define a $\rho^*(k)$-invariant metric d on $\mathcal{E}_S^*(k)$ by:

$$d(a, b)^2 := H^*(a - b, a - b)$$

$$||v|| := d(v, 0)$$

For any $\epsilon > 0$ denote the ball with radius ϵ centered at $a \in \mathcal{E}_S^*(k)$ by

$$B_d(\epsilon, a) := \{b \in \mathcal{E}_S^*(k) \mid d(a, b) < \epsilon\}$$

We call a point $y \in X_r^*(p)$ *ϵ-wide* if it has a neighborhood X_y such that ev_r^* maps X_y biholomorphically onto the ball $B_d(\epsilon, ev_r^*(y))$. We will see that there exists an $\epsilon > 0$ such that *every* $y \in X_r^*(p)$ is $(||ev_r^*(y)|| \cdot \epsilon)$-wide. To find such an ϵ consider for each $N \in \{1, 2, \ldots\}$ the following set:

$$X_N = \{x \in X_r^*(p) \mid x \text{ is } \delta\text{-wide, for some } \delta > \frac{||\mathrm{ev}_r^*(x)||}{N}\}$$

Then one easily checks:

1. X_N is open for all N.
2. If $N \leq M$ then $X_N \subseteq X_M$.
3. Each X_N is $\mathrm{Aut}(X_r^*(p)|\mathbb{C}^n \backslash \{0\})$ invariant and projects onto a weighted $\mathbb{C}^*$ invariant subset of $\mathbb{C}^n \backslash \{0\}$.
4. Each $x \in X_r^*(p)$ is contained in some X_N.

Observations 1, 3 and 4 imply that the projections of the sets X_N form a covering of $\mathbb{P}_d(\mathbb{C}^n)$ with open sets. The space $\mathbb{P}_d(\mathbb{C}^n)$ being compact, this implies that $X_r^*(p)$ is already covered by finitely many sets $X_{N_1}, \ldots, X_{N_m}$. Now 2 implies that $X_r^*(p) = X_N$ for some $N \in \{1, 2, \ldots\}$. Then we can take $\epsilon = 1/N$. It follows that if we have an inverse for ev_r^* on some neighborhood of $a \in \mathcal{E}_S^*(k)$, then this local inverse automatically extends to an inverse of ev_r^* on at least $B_d(\epsilon||a||, a)$. Hence every local inverse can be extended holomorphically to all of $\mathcal{E}_S^*(k)\backslash\{0\}$ because this is a simply connected set.

This in turn implies that ev_r^* is globally injective, because $\{x \in X_r^*(p) \mid \mathrm{ev}_r^*(x) \neq 0\}$ is connected. Now ev_r^* cannot attain the value 0, for suppose $\mathrm{ev}_r^*(x) = 0$, then ev_r^* would be constant on the π_r^* fibre containing x, violating the injectivity of ev_r^*. This proves that ev_r^* maps $X_r^*(p)$ biholomorphically onto $\mathcal{E}_S^*(k)\backslash\{0\}$.

Let ϕ be a holomorphic inverse of ev_r^* on $\mathcal{E}_S^*(k)\backslash\{0\}$. The composition $\pi_r^* \circ \phi$ can be extended to $\mathcal{E}_S^*(k)$ (Hartog) revealing $\mathcal{E}_S^*(k)$ as the universal branched covering of $\mathbb{C}^n$ branching with the prescribed indices along the subregular points. This clearly proves theorem 3.14.□

We repeat the important observation at the end of the proof in the next theorem.

Theorem 3.15 *If the marked Coxeter diagram is of elliptic type, the multivalued mapping* $\mathrm{Rev}(k)$ *has a single valued inverse* $\pi_r : \mathcal{E}_S^*(k) \to \mathbb{C}^n$*. Moreover,* π_r *is the universally branched covering branching along the subregular points with the prescribed indices* p_j*.*

Proof: □

We can now easily draw some remarkable consequences from this theorem. The following facts were already known, but proofs for corollaries 3.6 [C] and 3.7 [OS] where only provided by (non-trivial) case by case checkings using a computer.

Corollary 3.5 *If a marked Coxeter diagram is elliptic, the associated complex reflection group is finite. Let z be the order of the center of W and ν the exponent of the marked diagram. Then z/ν is an integer and the order of the complex reflection group equals $|W|\nu^{-n}$.*

Proof: From (weighted) homogeneity of the covering π_r we conclude that it is finite (it is locally finite at $0 \in \mathcal{E}_S^*(k)$). The degrees of π_r are d_j/ν, $1 \leq j \leq n$. This shows that z/ν is an integer because $z = \gcd(d_1, \ldots, d_n)$. The order of a reflection group is the product of its degrees. Hence the order of the complex reflection group equals $|W|\nu^{-n}$. $\square$

Corollary 3.6 (Coxeter) *A finite reflection group associated with an elliptic connected marked Coxeter diagram has the following presentation:*

$$\begin{array}{lll} \langle r_1, \ldots, r_n & | & r_j^{p_j} = e, \ j \in \{1, \ldots, n\} \\ & & (r_i, r_j)^{m_{ij}} = (r_j, r_i)^{m_{ji}} \ 1 \leq i < j \leq n \rangle \end{array}$$

Here the m_{ij} denote the Coxeter integers of the diagram

Proof: Such a group is just the group of automorphisms of the universally ramified covering, hence isomorphic to a braid group modulo order relations. $\square$

Corollary 3.7 (Orlik & Solomon) *The primitive homogeneous invariants*

$$Q_1, \ldots, Q_n \in P[\mathbb{C}^n]^G$$

of a finite complex reflection group G associated with an elliptic marked Coxeter diagram, can be chosen in such a way that the mapping $(Q_1, Q_2, \ldots, Q_n)$ is a ramified covering of $\mathbb{C}^n$ with branch locus Δ.

Proof: Just note that the covering mapping π_r is a weighted homogeneous polynomial mapping. Hence its coordinates are primitive homogeneous invariants for the reflection group G.$\square$

3.6 The parabolic case

In this section we will assume that the marked diagram (M, p) is of parabolic type, i.e. $\nu = 0$ and M has rank n. This implies also that all connected subdiagrams are of elliptic type. The $\mathbb{C}^*$-action on X lifts to a $\mathbb{C}$-action on $\widetilde{X}$ according to the commuting diagram in figure 3.2.

This action is *free*, indeed $x \mapsto 1 \cdot x$ is just the action of the central element $(g_1 g_2 \cdots g_n)^h$ on $\widetilde{X}$, which is not of finite order. Because all subdiagrams are elliptic, there exists a universally branched covering $\pi_r^* : X_r^*(p) \to \mathbb{C}^n \setminus \{0\}$ The

$$
\begin{tikzcd}
\mathbb{C} \times \widetilde{X} \arrow[r, "\cdot"] \arrow[d, "\exp(2\pi i \cdot) \times \pi"'] & \widetilde{X} \arrow[d, "\pi"'] \\
\mathbb{C}^* \times X \arrow[r, "\cdot"] & X
\end{tikzcd}
$$

Figure 3.2: $\mathbb{C}$-action on $\widetilde{X}$.

$$
\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{ev}"] \arrow[d] & A(k) \\
X_u(p) \arrow[r] \arrow[ur, "ev_u"] \arrow[d] & X_r^*(p) \arrow[u, "ev_r^*"'] \arrow[d] \\
X \arrow[r] & \mathbb{C}^n \setminus \{0\}
\end{tikzcd}
$$

Figure 3.3: The parabolic case.

$\mathbb{C}$-action on $\widetilde{X}$ induces a $\mathbb{C}$-action on $X_r^*(p)$. As in the elliptic case, we can lift the evaluation mapping to a locally biholomorphic mapping ev_r^* on $X_r^*(p)$. Hence we obtain the diagram in figure 3.3.

We pick a basis $e_1(k), \ldots, e_n(k), \ell(k)$ of $\mathcal{F}_S(k)$ as in section 2, and denote the dual basis by $e_1^*, \ldots, e_n^*, \ell^*$. (So

$$\rho(k, g_i)e_j(k) = e_j(k) + s_{ij}e_i(k)$$

etc.) One checks that the evaluation mapping satisfies

$$ev(k, \lambda \cdot x) = ev(k, x) + \log \lambda \cdot \ell^*$$

for all $x \in X$, $\lambda \in \mathbb{C}^*$. Hence the map ev_r^* satisfies

$$ev_r^*(\lambda \cdot x) = ev_r^*(x) + 2\pi i \lambda \cdot \ell^*$$

for all $\lambda \in \mathbb{C}$, $x \in X_r^*(p)$. In particular, the $\mathbb{C}$-action on $X_r^*(p)$ is *free.*

To prove our main result, we need a $\rho^*(k)$-invariant metric on the affine space $A(k)$ introduced in section 3.4.

Lemma 3.6 *Let $A_o(k) \subset \mathcal{F}_S^*(k)$ denote the annihilator of the constant functions. There exists a basis $v_1, \ldots, v_n$ of $A_o(k)$ such that:*

$$\rho^*(g_i)v_j = v_j + s_{ji}v_i, \text{ For all } 1 \leq i, j \leq n$$

Proof: Define v_i as $s_i\ell^* + \sum s_{ij}e_j^*$. Then one checks that these v_i lie in $A_o(k)$ and satisfy the stated identities. Remains to prove that they are independent.

By lemma 3.2 every non-trivial invariant subspace of $A_o(k)$ contains $\mathbb{C}\ell^*$. The vectors v_i span such a space and hence ℓ^* is a linear combination of the v_i. But the span of v_i does not equal $\mathbb{C}\ell^*$ and must therefore be at least n-dimensional (again by lemma 3.2). This proves that the v_i are independent.□

By this theorem we conclude that there exists a ρ^* invariant hermitian form $H^\times$ on $A_o(k)$. Moreover, $H^\times$ can be chosen *parabolic*. We now define the "metric" on $A(k)$ and the corresponding "balls" by:

$$d(a,b)^2 = H^\times(a-b, a-b), \quad a, b \in A(k)$$

$$B_d(\epsilon, a) = \{b \in A(k) \mid d(b,a) < \epsilon\}, \quad a \in A(k),\ \epsilon > 0$$

Note that these balls actually are *tubes*. They are invariant under translation along any multiple of ℓ^*.

We can now state and prove the main result.

Theorem 3.16 *The mapping* ev_r^* *maps* $X_r^*(p)$ *biholomorphically onto* $A(k)$.

Proof: Analogous to the elliptic case. We call a point $x \in X_r^*(p)$ ϵ-wide if there exists a neighborhood Y_x of x such that ev_r^* maps Y_x biholomorphically onto $B_d(\epsilon, \mathrm{ev}_r^*(x))$. The claim is that there exists an $\epsilon > 0$ such that every point of $X_r^*(p)$ is ϵ-wide. Consider for each $N \in \mathbb{N}^*$ the following set:

$$X_N = \{x \in X_r^*(p) \mid x \text{ is } \delta\text{-wide for some } \delta > 1/N\}$$

Again these sets satisfy the following properties:

1. Each X_N is an open set.
2. Each X_N is $\mathbb{C}$ and $\mathrm{Aut}(X_r^*(p)|\mathbb{C}^n\backslash\{0\})$ invariant.
3. If $N \leq M$, then $X_N \subseteq X_M$.
4. Every $x \in X_r^*(p)$ is contained in some X_N.

Only statement 4 needs some extra explanation. It follows by combining the fact that ev_r^* is locally biholomorphic and its transformation behaviour w.r.t. the $\mathbb{C}$-action on $X_r^*(p)$. Now statements 1, 2 and 4 imply that the sets X_N form a covering of the compact space $\mathbb{P}_d(\mathbb{C}^n)$ with open sets. From 3 we conclude that $X_r^*(p) = X_N$ for some $N \in \mathbb{N}^*$. Hence every point of $X_r^*(p)$ is ϵ-wide if we take $\epsilon = 1/N$.

Now every local inverse of ev_r^* at $ev_r^*(x)$ can be extended to at least the tube $B_d(\epsilon, ev_r^*(x))$. Because $A(k)$ is simply connected we conlude that ev_r^* admits a holomorphic inverse on all of $A(k)$. This proves the theorem. □

To deduce a presentation for the geometric realisation $G(M,p)$ we need the following lemma.

Lemma 3.7 *View the reflection representation $\varrho(k)$ of $B(M,p)$ as a $2n$ dimensional representation over $\mathbb{R}$. Then the only non-trivial invariant subspaces (over $\mathbb{R}$) are contained in $\mathbb{C}\lambda$ where λ denotes a non-zero $\varrho(k)$-fixed vector (unique upto a complex scalar).*

Proof: Let U be an invariant subspace (over $\mathbb{R}$), $U \neq \{0\}$. The endomorphism $1-\varrho(k,g_j)$ maps into $U \cap \mathbb{C}e_j$. Suppose U is not contained in $\mathbb{C}\lambda$ then we can assume $c_r e_r \in U$ for some $r \in \{1,\ldots,n\}$ and some $c_r \in \mathbb{C}^*$. Now let $j \in \{1,\ldots,n\}$ be arbitrary. Because e_r is a cyclic vector for $\varrho(k)$ (over $\mathbb{C}$) there is a $g \in B(M,p)$ such that $(1-\varrho(k,g_j))\varrho(k,g)(c_r e_r) \neq 0$. But this implies that we may assume $c_j e_j \in U$ for some $c_j \in \mathbb{C}^*$.

Now for any i,j we have

$$(1-\varrho(k,g_j))(1-\varrho(k,g_i))(c_j e_j) = c_j s_{ij} s_{ji} e_j \in U$$

If i,j are chosen in such a way that $m_{ij} > 2$ and not both q_i and q_j equal 1, then $s_{ij}s_{ji}$ is not a real number. This implies that $\mathbb{C}e_j \subseteq U$ and consequently $U = \mathbb{C}^n$. □

Corollary 3.8 *The geometric realisation $G(M,p)$ of $B(M,p)$ has the following presentation:*

$$\begin{array}{lll} \langle r_1, r_2, \ldots, r_n & | & r_i^{p_i} = e, \ i \in \{1,\ldots,n\} \\ & & (r_i, r_j)^{m_{ij}} = (r_j, r_i)^{m_{ji}}, \ 1 \leq i < j \leq n \\ & & (r_1 r_2 \cdots r_n)^{h/z} = e \rangle \end{array}$$

Proof: The geometric realisation as a matrix-representation is equivalent to the restriction of $\rho^*(k)$ to $A_o(k)$. The matrixgroup generated by $\rho^*(k)$ on $\mathcal{F}_S^*(k)$ is isomorphic to $B(M,p)$ according to the previous theorem. The kernel of the homomorphism "restriction to $A_o(k)$" consists exactly of all elements acting as a *translation* on $A(k)$. The set of all occuring translation vectors in $A_o(k)$ is a *discrete* abelian subgroup of $A_o(k)$, denoted by L. The set L is clearly $\rho^*(k)$-invariant. Hence by the previous lemma, L is either of rank $2n$, or contained in $\mathbb{C}\ell^*$. However L cannot be of full rank, for this would imply that $\mathbb{C}^n \backslash \{0\}$ is compact (being a quotient of $A(k)/L$).

We conclude that L must be contained in $\mathbb{C}\ell^*$. Moreover, by considering the $\mathbb{C}$-action on $X_r^*(p)$ one finds $L = \mathbb{Z}\frac{2\pi i}{z}\ell^*$. The kernel of the restriction is generated

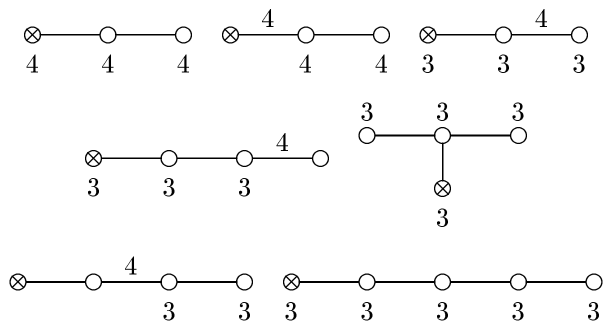

Table 3.1: The seven parabolic diagrams.

by $\rho^*(k, g_1 g_2 \cdots g_n)^{h/z}$. Hence the presentation of $B(M,p)$ has to be extended by one relation exactly as stated in the corollary. $\square$

We conclude this section by deducing a Chevalley theorem on the invariants in certain rings of theta functions. The results are similar to those obtained by Looijenga in [L]. Because the parabolic cases for which n equals two are directly related to the classical theory of the Gauss function, we will restrict ourselves to the study of the seven remaining parabolic cases, listed in table 3.1. (In the diagram the first vertex is indicated by a cross mark.)

In each case, monodromy induces a transformation group $C(M,p)$ of the $(n-1)$-dimensional affine space $A_\ell := A(k)/\mathbb{C}\ell^*$. This group acts discretely, cocompactly and is generated by n affine complex reflections satisfying the order and braid relations as indicated by the marked Coxeter diagram (M,p). The reflections $\rho^*(k,g_2),\ldots,\rho^*(k,g_n)$ have a unique simultaneous fixed point on A_ℓ which we will denote by f. Note that f can be taken a scalar multiple of e_1^* (mod ℓ^*). We will study this later on.

Observe that $H^\times$ really induces a *metric* on A_ℓ. Introduce the *point group* P of $C(M,p)$ as certain isometries of A_ℓ fixing f as follows. The group P will be the image of the homomorphism

$$p : C(M,p) \to \mathrm{Aut}(A_\ell),\ p(g) : v \mapsto g(v) - g(f) + f$$

Then $p(g)$ fixes f and p is indeed a homomorphism. We also write p for the pull back of p to $B(M)$ by $\rho^*(k)$. Note that P is generated by $p(g_1),\ldots,p(g_n)$ and these transformations are again complex reflections satisfying the order and braid relations of (M,p).

Denote the translation of A_ℓ over $\lambda \in A_o(k)/\mathbb{C}\ell^*$ by t_λ and take

$$\Lambda = \{\lambda \in A_o(k)/\mathbb{C}\ell^* \mid t_\lambda \in C(M,p)\}$$

If $\lambda \in \Lambda$ then $g(f+\lambda) - f \in \Lambda$ for all $g \in P$. Indeed if $p(g_o) = g$ for $g_o \in C$ then $t_{g(f+\lambda)-f} = g_o t_\lambda g_o^{-1}$. The point group acts naturally on $A_o(k)/\mathbb{C}\ell^*$ and stabilizes Λ. Because $C(M,p)$ acts discretely on A_ℓ and P acts irreducibly even over $\mathbb{R}$ (only trivial P-stable affine subspaces) we conclude that Λ is either $\{0\}$ or a *lattice*.

We can now prove the following important theorem:

Theorem 3.17 *The group $C(M,p)$ is the semidirect product of its normal translation subgroup T_Λ and its point group P: $C(M,p) = T_\Lambda P$. The group P is isomorphic to the complex reflection group associated to the subdiagram of M obtained by deleting the first node. Moreover Λ is a lattice of the form*

$$\Lambda = Span_{\mathbb{Z}}\{\rho^*(k,g)\lambda \mid g \in B(M)\}$$

for some special eigenvector $\lambda \in A_o(k)/\mathbb{C}\ell^*$ of $\rho^*(k,g_1)$.

Proof: The subgroup of P given by $\langle p(g_2),\ldots,p(g_n)\rangle$ is isomorphic to the reflection group $\langle r_2(k),\ldots,r_n(k)\rangle$ acting on $\mathbb{C}^{n-1}$. One computes that in all seven parabolic cases this reflection group already contains a complex reflection r satisfying the same order and braid relations as $p(g_1) \in P$. Because $\langle r_2(k),\ldots,r_n(k)\rangle$ fixes a positive definite hermitean structure on $\mathbb{C}^{n-1}$ it follows that

$$p(g_1) \in \langle p(g_2),\ldots,p(g_n)\rangle.$$

Indeed the relations imply an explicit expression of such a reflection in terms of the hermitean structure.

Now compute

$$p(g_1^{-1})\rho^*(k,g_1)v = v + (1-\rho^*(k,g_1^{-1}))f.$$

So $p(g_1^{-1})\rho^*(k,g_1)$ is a translation over a non zero special eigenvector λ of $\rho^*(k,g_1)$.

The statements of the theorem now follow from the remarks that $p(g_1^{-1}) \in C(M,p)$ and $C(M,p)$ is generated by $p(g_2),\ldots,p(g_n)$ and $t_{(1-\rho^*(k,g_1^{-1}))f}$. $\square$

Remark 3.4 *It turns out that the two crystallographic groups*

$$C(A_3,4) \text{ and } C(B_3,4,2)$$

are isomorphic. *In both cases the point group is isomorphic to $B(A_2,4)$ and the lattice is generated by a special eigenvector of $\rho^*(k,g_2)$.*

Remark 3.5 *A complete classification of complex crystallographic reflection groups can be found in an article by Popov [P].*

The next step is now to introduce a certain kind of theta functions on A_ℓ. Let the inverse of the evaluation mapping on $A(k)$ be given by:

$$\phi = (\phi_1, \ldots, \phi_n) : A(k) \to \mathbb{C}^n \backslash \{0\}$$

By using some properties of the evaluation mapping one deduces for all j in $\{1, \ldots, n\}$:

1. $\phi_j(u + x\ell^*) = e^{d_j x}\phi_j(u),\ u \in A(k),\ x \in \mathbb{C}$. ($d_j$ is the j^{th} invariant degree of the real reflection group W).
2. $\phi_j(\rho^*(k,g)u) = \phi_j(u)$ for all $u \in A(k)$ and $g \in B(M,p)$.

Let $\beta \in \mathcal{F}_S(k)$ be such that $\ell^*(\beta) = 1$ and $\rho(k, g_j)\beta = \beta$ for $j = 2, \ldots, n$. Such a β is unique modulo the constant functions. Consider the entire function θ_j on A_ℓ defined by

$$\theta_j(u + \mathbb{C}\ell^*) = e^{-d_j u(\beta)}\phi_j(u).$$

Using the properties of ϕ_j one checks that θ_j is well defined and satisfies:

$$\theta_j(\rho^*(k,g)u) = e^{-d_j(u(\rho(k,g)\beta) - u(\beta))}\theta_j(u),\ g \in B(M,p)$$

In particular $\theta_j(gu) = \theta_j(u)$ for all $u \in A_\ell, g \in P$. From these transformation formulae we see that θ_j is a P-invariant theta function on A_ℓ with respect to the lattice Λ. Let us now study the general theory of such theta functions. In each of the seven parabolic cases there exists a unique $\rho^*(k)$-invariant positive definite Hermitean structure $(\cdot,\cdot)$ on $A_o(k)/\mathbb{C}\ell^*$ satisfying $\mathrm{Im}(\Lambda,\Lambda) = \mathbb{Z}$. The alternating form $\mathrm{Im}(\cdot,\cdot)$ turns out to be non-degenerate. It is well known [SD] that there exists a basis of Λ over $\mathbb{Z}$ such that the matrix of this alternating form with respect to this basis takes the following form:

$$\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

Here d is a diagonal matrix $\mathrm{diag}(t_1, \ldots, t_{n-1})$ for some positive integers satisfying $1 = t_1 \mid t_2 \mid \ldots \mid t_{n-1}$. These integers are called the *invariant factors* of the alternating form. The invariant factors are listed in table 3.2.

Theorem 3.18 *Suppose ϑ is a theta function on A_ℓ satisfying:*

1. *$\vartheta(u + \lambda) = e^{L(u,\lambda)}\vartheta(u)$ for all $u \in A_\ell$ and $\lambda \in \Lambda$. Here $L(\cdot,\lambda)$ is an* affine *function for all λ.*
2. *The function $u \mapsto \vartheta(gu)$ transforms as stated in 1 for all $g \in P$.*

There is a unique $\epsilon : \Lambda \to \mathbb{Z}/2\mathbb{Z}$, independent of ϑ, and a $D \in \mathbb{N}$ such that

1. *$L(u,\lambda) \equiv D(\pi(u-f,\lambda)+\frac{\pi}{2}(\lambda,\lambda)+\pi i\epsilon(\lambda))$ $(mod\ 2\pi i)$, for all $u \in A_\ell, \lambda \in \Lambda$.*
2. *$\epsilon(\lambda+\mu) \equiv \epsilon(\lambda)+\epsilon(\mu)+Im(\lambda,\mu)$ $(mod\ 2)$, for all $\lambda,\mu \in \Lambda$.*

*Here f denotes the P-fixed point in A_ℓ. We call such a ϑ a P-*stable *theta function of* degree D.

Proof: This relies heavily on the general theory of theta functions. See for example [SD].

Clearly $L(u,\lambda)$ must be of the form $(u-f,L\lambda)+Q(\lambda)$ for some *real* linear transformation L of $A_o(k)/\mathbb{C}\ell^*$. By P invariance one deduces that L commutes with all $\rho^*(k,g_j)$, $j \in \{2,\dots,n\}$. This implies that L is a (complex) scalar multiplication. Hence there is a $D \in \mathbb{C}$ such that $L(u,\lambda) = D\pi(u-f,\lambda)+Q(\lambda)$. From the cocycle relation for $L(u,\lambda)$ it follows that in fact $D \in \mathbb{Z}$.

It is now general theory of theta functions that shows that $L(u,\lambda)$ must be of the form

$$L(u,\lambda) \equiv D(\pi(u-f,\lambda)+\frac{\pi}{2}(\lambda,\lambda)+\pi i\epsilon(\lambda)) \pmod{2\pi i}$$

for some P-invariant $\epsilon : \Lambda \to \mathbb{C}/2\mathbb{Z}$ satisfying the relation stated in the theorem. From the explicit form of Λ and $(\cdot,\cdot)$ one can check that in all seven parabolic cases the function ϵ is uniquely determined and takes values in $\mathbb{Z}/2\mathbb{Z}$. □

Corollary 3.9 *For each degree d_j there is a $D_j \in \mathbb{N}_{\geq 1}$ such that if $\rho^*(k,g)$ induces $t_\lambda \in C$ then*

$$-d_j(u(\rho(k,g)\beta)-u(\beta)) \equiv D_j(\pi(u-f,\lambda)+\frac{\pi}{2}(\lambda,\lambda)+\pi i\epsilon(\lambda))\ (mod\ 2\pi i)$$

for all $u \in A_\ell$.

Proof: The theta function θ_j is P-stable and transform under translation over λ by the exponential of the left hand side of this equality. Hence by the previous theorem there exists a D_j as stated. □

Note that the degree of θ_j equals D_j. The degrees $D_1,\dots,D_n$ are listed in table 3.2.

Let Θ_D be the set of P-stable theta functions of degree D. For all D, Θ_D is a finite dimensional $\mathbb{C}$ vector space. In fact it the dimension of Θ_D equals D^{n-1} times the product of the invariant factors of the alternating form $\mathrm{Im}(\cdot,\cdot)$ [SD]. Let

$$\Theta = \bigoplus_{D\geq 0} \Theta_D$$

then Θ is a graded $\mathbb{C}$ algebra. The point group P acts naturally on this algebra. Note that the algebras of P-stable theta functions are *isomorphic* for the two cases $(A_3,4)$ and $(B_3,4,2)$.

We denote the subalgebra of P-invariant theta functions by Θ^P.

Theorem 3.19 *For all parabolic groups except* $(B_3, 4, 2)$ *the algebra* Θ^P *equals* $\mathbb{C}[\theta_1, \ldots, \theta_n]$. *In particular it is isomorphic to a* polynomial algebra.

Proof: We do not consider the marked diagram $(B_3, 4, 2)$ for it turns out that $(A_3, 4)$ determines the invariants in Θ for that case.

As coordinates of the inverse of the evaluation mapping it is clear that the ϕ_j and hence the θ_j are algebraically independent. If $\vartheta \in \Theta^P$ is of degree D then consider the function $\overline{\vartheta} : A(k) \to \mathbb{C}$ defined by

$$\overline{\vartheta}(u) = e^{Du(\beta)}\vartheta(u + \mathbb{C}\ell^*)$$

One checks that it satisfies

1. $\overline{\vartheta}(u + x\ell^*) = e^{xD}\overline{\vartheta}(u)$ for all $u \in A(k), x \in \mathbb{C}$.
2. $\overline{\vartheta}(\rho^*(k, g)u) = \overline{\vartheta}(u)$ for all $g \in B(M, p)$.

Note that by P-invariance of ϑ it suffices to check 2 for all g such that $\rho^*(k, g)$ induces a translation t_λ of C. To check this use corollary 3.9 and the degrees $D_1, \ldots, D_n$ as listed in the table.

Using these properties it follows that the composition $\overline{\vartheta} \circ ev(k)$ extends to a weighted homogeneous polynomial of degree D on $\mathbb{C}^n$. Hence ϑ is a polynomial in $\theta_1, \ldots, \theta_n$. $\Box$

Remark 3.6 *Similarly one can prove that the algebra of invariants of* even degree *in* Θ^P *related to* $C(B_3, 4, 2)$ *also equals* $\mathbb{C}[\theta_1, \ldots, \theta_n]$. *(Here the* $\theta_1, \ldots, \theta_n$ *are the theta functions related to the diagram* $(B_3, 4, 2)$*).*

To end this section I give a sketch of the method to compute the degrees D_j. Recall that $\rho(k, g_j)\ell = \ell + x_j e_j$ where the constants x_j are chosen in such a way that

$$x_1\rho(k, g_2 \cdots g_n)e_1 + x_2\rho(k, g_3 \cdots g_n)e_2 + \ldots + x_n e_n = \frac{2\pi i}{h}$$

where the right hand side is a constant function. In particular it is the (upto a scalar) unique monodromy fixed vector. From this we can explicitly compute β.

Take $y_2, \ldots, y_n \in \mathbb{C}$ such that

$$x_j + \sum_{l=2}^{n} y_l s_{jl} = 0$$

for all $j \in \{2, \ldots, n\}$. Projection of ℓ onto the $\rho(k, g_2), \ldots, \rho(k, g_n)$ fixed vectors along the span of $e_2, \ldots, e_n$ gives

$$\beta = \ell + \sum_{j=2}^{n} y_j e_j.$$

Diagram	Invariant factors of $\mathrm{Im}(\cdot,\cdot)$	Invariant degrees	(λ,λ)
$(A_3,4)$ and $(B_3,2,4)$	$1,2$	$2,3,4$	2
$(B_3,3,3)$	$1,6$	$1,2,3$	$2\sqrt{3}$
$(B_4,3,2)$	$1,3,3$	$1,2,3,4$	$2\sqrt{3}$
$(D_4,3)$	$1,3,3$	$1,2,2,3$	$2\sqrt{3}$
$(F_4,2,3)$	$1,1,3$	$1,3,4,6$	$\frac{4}{3}\sqrt{3}$
$(A_5,3)$	$1,1,3,3$	$2,3,4,5,6$	$2\sqrt{3}$

Table 3.2: Structure of the parabolic groups.

Applying e_1^* to $2\pi i/h$ yields that we can take

$$f = \frac{2\pi i}{hx_1}e_1^* + \mathbb{C}\ell^*$$

for the $\rho^*(k,g_2),\dots,\rho^*(k,g_n)$ fixed vector in A_ℓ.

Consider t_λ with $\lambda = (1-\rho^*(k,g_1^{-1}))f$ a generator of Λ as before. Then t_λ is induced by $\rho^*(k,gg_1)$ for some g in $\langle g_2,\dots,g_n\rangle$. By corollary 3.9 we know

$$-d_j(f(\rho(k,gg_1)\beta)-f(\beta)) \equiv D_j(\frac{\pi}{2}(\lambda,\lambda)+\pi i\epsilon(\lambda)) \pmod{2\pi i}$$

The real part of the right hand side can be computed from table 3.2 where (λ,λ) is listed for each case. Substituting all explicit formulas in the left hand side and considering the fact that $\rho(k,g)\beta=\beta$ we get:

$$-d_j\frac{2\pi i}{hx_1}(x_1+\sum_{j=2}^{n}y_js_{1j}) \equiv D_j(\frac{\pi}{2}(\lambda,\lambda)+\pi i\epsilon(\lambda)) \pmod{2\pi i}.$$

With this result the degrees D_j can be computed in each case.

3.7 The hyperbolic case

Throughout this section we assume that the connected marked diagram (M,p) is of hyperbolic type. This means that if we define $k\in K$ by $k_j := 1/2-1/p_j$

then $\nu(k)$ satifies $1 - m_2 < \nu(k) < 0$. Lift $Rev(k)$ to a single valued mapping $\widetilde{ev}$ on $\widetilde{X}$. Then $\widetilde{ev}$ is a locally biholomorphic mapping satisfying:

$$\widetilde{ev}(x \cdot y) = e^{2\pi i \nu x} \cdot \widetilde{ev}(y), \text{ for all } x \in \mathbb{C},\ y \in \widetilde{X}$$

by homogeneity of $Rev(k)$. Again we define an invariant Hermitian form H^* on $\mathcal{E}_S^*(k)$, i.e. the signature of H^* is $(1, n-1)$.

Definition 3.14 *The set of vectors in $\mathcal{E}_S^*(k)$ on which H^* is positive is denoted by $\mathbb{B}$. The unit ball in $\mathbb{C}^{n-1}$ by B. In a formula:*

$$\mathbb{B} = \{v \in \mathcal{E}_S^*(k) \mid H^*(v,v) > 0\}$$

$$B = \{(x_1, \dots, x_{n-1}) \in \mathbb{C}^{n-1} \mid |x_1|^2 + \dots + |x_{n-1}|^2 < 1\}$$

Lemma 3.8 *The set $\mathbb{B}$ is a trivial $\mathbb{C}^*$-bundle over B. To be precise: there is a biholomorphic mapping*

$$\tau : \mathbb{B} \to \mathbb{C}^* \times B$$

such that if $\tau(v) = (x, \mu)$ then for all $\zeta \in \mathbb{C}^*$, $\tau(\zeta v) = (\zeta x, \mu)$.

Proof: Let $\xi_1, \dots, \xi_n$ be a basis of $\mathcal{E}_S^*(k)$ such that:

$$H^*(\xi_i, \xi_j) = \pm\delta_{ij},\ H^*(\xi_n, \xi_n) = 1$$

Note that if $v \in \mathbb{B}$ then the ξ_n coordinate of v (i.e. $H^*(v, \xi_n)$) is non-zero. This allows the following construction of τ:

$$\tau : \mathbb{B} \to \mathbb{C}^* \times B,\ \tau : \sum_{j=1}^{n} c_j \xi_j \mapsto (c_n, \frac{c_1}{c_n}, \dots, \frac{c_{n-1}}{c_n})$$

One easily checks that this mapping satisfies the presumed conditions. $\Box$

Corollary 3.10 *The fundamental group of $\mathbb{B}$ is isomorphic to $\mathbb{Z}$, moreover*

$$\varpi : \tau^{-1} \circ (exp(2\pi i \cdot) \times id) : \mathbb{C} \times B \to \mathbb{B}$$

is a universal covering of $\mathbb{B}$.

Proof: Evident. $\Box$

The following theorem is fundamental for the hyperbolic theory. However, because the proof of it would be a little distracting at this moment, I put it in the seperate section 3.8.

Theorem 3.20 *In case (M,p) is of hyperbolic type, the image of the associated multivalued mapping $\mathrm{Rev}(k)$ is contained in $\mathbb{B}$.*

Proof: In section 3.8. □

Because $\widetilde{X}$ is simply connected, we can factor the map $\widetilde{ev}$ through the universal covering of $\mathbb{B}$. In this way, we get a mapping

$$\widetilde{\mathrm{EV}} : \widetilde{X} \to \mathbb{C} \times B$$

satisfying $\widetilde{ev} = \varpi \circ \widetilde{\mathrm{EV}}$. Now $\rho^*(k)$ induces a unique group $\widetilde{G}$ of transformations of $\mathbb{C} \times B$ and surjective homomorphisms

$$\tilde{\rho} : B(M) \to \widetilde{G}, \ \mathrm{pr} : \widetilde{G} \to \rho^*(k, B(M))$$

such that for every $g \in B(M)$ we obtain the commuting diagram in figure 3.4.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{\mathrm{EV}}"] \arrow[d, "g"'] & \mathbb{C} \times B \arrow[r, "\varpi"] \arrow[d, "\tilde{\rho}(g)"'] & \mathbb{B} \arrow[d, "\rho^*(k,g)"'] \\
\widetilde{X} \arrow[r, "\widetilde{\mathrm{EV}}"] & \mathbb{C} \times B \arrow[r, "\varpi"] & \mathbb{B}
\end{tikzcd}$$

Figure 3.4: The $\widetilde{G}$-action.

Denoting the $\mathbb{C}$-action $(y, \mu) \mapsto (y + x, \mu)$ on $\mathbb{C} \times B$ by $x \cdot (y, \mu)$, the mapping $\widetilde{\mathrm{EV}}$ also satisfies:

$$\widetilde{\mathrm{EV}}(x \cdot y) = \nu x \cdot \widetilde{\mathrm{EV}}(y), \ x \in \mathbb{C}, \ y \in \widetilde{X}$$

Let $\Delta_\infty \subset \Delta$ be the union of facets associated to non elliptic connected subdiagrams of (M, p). In particular $0 \in \Delta_\infty$. Denote the universal degree p covering by $\pi_u : X_u(p) \to X$.

Lemma 3.9 *The mapping $\widetilde{EV}$ descends to a locally biholomorphic mapping ev_u on $X_u(p)$.*

Proof: The fact that $Rev(k)$ maps into $\mathbb{B}$ together with theorem 3.13 implies that $Rev(k)$ maps some small neighborhood in X of a subregular point in Δ into some simply connected open sub set of $\mathbb{B}$. (The image cannot wrap around the origin.) This implies that $\widetilde{\mathrm{EV}}$ is invariant under continuation along any p_j-fold loop around a type j reflection plane. Hence $\widetilde{\mathrm{EV}}$ descends to $X_u(p)$. □

Corollary 3.11 *The homomorphism $\tilde{\rho}$ projects to a homomorphism*

$$\tilde{\rho} : B(M,p) \cong Aut(X_u(p)|X) \to \widetilde{G}$$

This describes the monodromy of ev_u*, i.e.*

$$\mathrm{ev}_u(g \cdot x) = \tilde{\rho}(g)\mathrm{ev}_u(x)$$

for all $g \in Aut(X_u(p)|X)$ *and all* $x \in X_u(p)$*.*

Proof: $\square$

Lemma 3.10 *The covering $X_u(p)$ can be embedded in a universally ramified covering $\pi_r^* : X_r^*(p) \to \mathbb{C}^n \backslash \Delta_\infty$. Moreover, ev_u extends to a locally biholomorphic mapping ev_r^* on $X_r^*(p)$.*

Proof: From the elliptic case we know that universally ramified extensions exist locally above any point of $\mathbb{C}^n \backslash \Delta_\infty$. By using properties of ev_u we can again conclude that all these local extensions fit together and obtain $X_r^*(p)$. By a similar argument as before, ev_u extends locally biholomorphically to $X_r^*(p)$.$\square$

We obtained the diagram in figure 3.5.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{EV}"] \arrow[d] & \mathbb{C} \times B \\
X_u(p) \arrow[ur, "\mathrm{ev}_u"] \arrow[r] \arrow[d] & X_r^*(p) \arrow[u, "ev_r^*"'] \arrow[d] \\
X \arrow[r] & \mathbb{C}^n \backslash \Delta_\infty
\end{tikzcd}$$

Figure 3.5: The hyperbolic case.

Before stating the main theorem of this section we investigate what happens near a facet in Δ_∞ associated to a connected sub diagram of (M,p) of parabolic type. So suppose (by renumbering) that the sub diagram spanned by the vertices $1, \ldots, j$ is connected and of parabolic type.

Lemma 3.11 *There exists a basis $e_1, \ldots, e_j, F, f_1, \ldots, f_{n-j-1}$ of $\mathcal{E}_S(k)$ such that*

1. *The vector e_l is a special eigenvector of $\rho(k, g_l)$ for all l.*

2. $\rho(k, g_1 g_2 \cdots g_j)F = F + 2\pi i \cdot f$. *Here* f *is a non zero vector in the* $\mathbb{C}$*-span of* $e_1, \ldots, e_j$ *such that* f *is fixed by all reflections* $\rho(k, g_1), \ldots, \rho(k, g_j)$.

3. *Every vector* f_l *is also fixed by all these* j *reflections.*

Proof: This follows from theorem 3.5 in section 3.2. $\square$

Pick a basis of $\mathcal{E}_S(k)$ as indicated and let $e_1^*, \ldots, e_j^*, F^*, f_1^*, \ldots, f_{n-j-1}^*$ denote the dual basis of $\mathcal{E}_S^*(k)$.

Lemma 3.12 *Take branches of* $Rev(k)$ *and* f *such that* $Rev(k,u)(f) = f(u)$. *Then*

$$\rho^*(k, g_1 \cdots g_j) Rev(k, u) = Rev(k, u) + 2\pi i \cdot f(u) F^*.$$

In particular $f(u) \neq 0$ *and* $f(u) H^*(\ell^*, Rev(u)) \in \mathbb{R}$.

Proof: This transformation formula follows from the fact that F is the only basis element that transforms non trivially under this partial Coxeter element. Because $Rev(k, u) \in \mathbb{B}$ we conclude that $f(u) \neq 0$ for there are no $\rho^*(k, g_1 \cdots g_j)$-fixed vectors in $\mathbb{B}$.

Write $||\mu||^2$ for $H^*(\mu, \mu)$. Then by monodromy invariance of H^* we get

$$||\rho^*(k, g_1 \cdots g_j)^t Rev(k, u)||^2 = ||Rev(k, u)||^2$$

for all $t \in \mathbb{N}$. Repeated application of the transformation formula from the lemma shows that $||F^*||^2 = 0$ and $f(u) H^*(F^*, Rev(k, u)) \in \mathbb{R}$ as stated. $\square$

Let p be a point on the facet under consideration. Then local monodromy near p fixes the vector F^* on the boundary of $\mathbb{B}$. To study the behaviour of $Rev(k)$ near p we use a local monodromy invariant distance function on $\mathbb{B}$ that measures the distance of a point to F^*. We define this distance for $v \in \mathbb{B}$ by:

$$\delta(v) = \frac{|(F^*, v)|^2}{(v, v)}$$

Note that it is constant on the line through v. If $\delta(v) \to 0$ then $v \to F^*$ in the projective sense.

Let $x_1, \ldots, x_n$ be local coordinates near p such that the facet is described by the equations $x_{j+1} = x_{j+2} = \ldots = x_n = 0$. Now consider

$$x_1, \ldots, x_j, y_1 := x_{j+1}, y_2 := \frac{x_{j+2}}{x_{j+1}}, \ldots, y_{n-j} := \frac{x_n}{x_{j+1}}$$

as local coordinates on the blow up of the facet. (So $y_1 = 0$ locally defines the exceptional divisor. The argument that follows does not depend on this particular choice of coordinates).

Lemma 3.13 *Each $e_1, \dots, e_j$ and $f_1, \dots, f_{n-j-1}$ extends holomorphically over the exceptional divisor. Moreover, in the coordinates x, y we can locally write*

$$F(x,y) = (\log(y_1) + \psi(x,y))f(x,y)$$

for some holomorphic ψ.

Proof: Recall that the exponent along the exceptional divisor is 0 with multiplicity $n+1$. By general theory of connections with regular singularities we know that

$$F(x,y) - \log(y_1)f(x,y), e_1, \dots, e_j, f_1, \dots, f_{n-j-1}$$

extend holomorphically over the divisor $y_1 = 0$. We already know that f does not vanish if $y_1 \neq 0$. Because the exponent of f along the exceptional divisor is 0, we conclude by Hartog's theorem that f is even non vanishing for $y_1 = 0$. Then we can clearly write F in the indicated form. □

Theorem 3.21 *Let $\mathrm{Rev}(k)$ and F be branches on the local coordinate neighborhood with coordinates (x,y) such that $\mathrm{Rev}(k,x,y)(F) = F(x,y)$. Then the distance $\delta(\mathrm{Rev}(k,x,y))$ tneds to 0 if y_1 tends to 0. Moreover, convergence is locally uniform w.r.t. the other coordinates.*

Proof: Write $Rev(k,x,y) = F(x,y)F^* + r(x,y)$. Then r extends holomorphically over the divisor $y_1 = 0$. Because $f(x,y)H^*(F^*, r(x,y)) \in \mathbb{R}$ and $f(x,y)$ is non vanishing even if $y_1 = 0$, we conclude that $H^*(F^*, r)$ is also non vanishing for $y_1 = 0$.

Now compute the distance (arguments (x,y) are omitted in the right hand side):

$$\delta(Rev(k,x,y)) = \frac{|H^*(F^*,r)|^2}{2\mathrm{Re}\Big(FH^*(F^*,r)\Big) + H^*(r,r)} =$$

$$= \frac{|H^*(F^*,r)|^2}{2\mathrm{Re}\Big((\log(y_1)+\psi)fH^*(F^*,r)\Big) + H^*(r,r)} =$$

$$= \frac{|H^*(F^*,r)|^2}{2fH^*(F^*,r)\Big(\log|y_1| + \mathrm{Re}(\psi)\Big) + H^*(r,r)}$$

Now $H^*(F^*,r) \neq 0$ so the logarithm in the denominator will cause converge of this distance as stated. □

Note that convergence is not only locally uniform, but also does not depend on the choice of the particular branches (δ is invariant under local monodromy). We can now prove the main theorem of this section.

Theorem 3.22 *If every connected proper sub diagram of (M, p) is either elliptic or parabolic, then the mapping*

$$\mathrm{ev}_r^* : X_r^*(p) \to \mathbb{C} \times B$$

is globally biholomorphic and onto.

Proof: We need again a $\widetilde{G}$ and $\mathbb{C}$ invariant "metric" on $\mathbb{C} \times B$. Consider the Poincaré-Bergman metric on $B \cong \mathbb{C}^* \backslash \mathbb{B}$:

$$\cosh d([v],[w]) = \frac{|H^*(v,w)|}{[H^*(v,v)H^*(w,w)]^{1/2}}$$

Now extend it trivially on the $\mathbb{C}$-fibres:

$$\delta((w_1,b_1),(w_2,b_2)) = d(b_1,b_2)$$

This "metric" is clearly $\widetilde{G}$ and $\mathbb{C}$ invariant. Define a ball (tube) w.r.t. this metric by:

$$B_\delta(\epsilon, x) = \{y \in \mathbb{C} \times B \mid \delta(x,y) < \epsilon\}$$

The proof of the similar theorem in the parabolic case has to be altered a little. We used that $\mathbb{P}_d(\mathbb{C}^n)$ is compact, but now we possibly left out some points by excluding Δ_∞. To overcome this problem we cover $\mathbb{C}^n \backslash \Delta_\infty$ in a certain way by closed sets. Let

$$K_1 \subset\subset K_2 \subset\subset K_3 \subset\subset \ldots$$

be a sequence of closed subsets of $\mathbb{C}^n \backslash \Delta_\infty$ such that

1. Each K_j is invariant under the weighted homogeneous $\mathbb{C}^*$ action.
2. Each set K_j is contained in the interior of K_{j+1}.

Then $\mathbb{C}^* \backslash K_j$ is a compact subset of $\mathbb{P}_d(\mathbb{C}^n)$.

Let $X_j \subset X_r^*(p)$ be the π_r^* pre image of K_j. Then there exists a sequence of positive numbers $\epsilon_1 \geq \epsilon_2 \geq \ldots$ such that any point of X_j is ϵ_j-wide (w.r.t. ev_r^*).

Suppose $\gamma : [0,1] \to \mathbb{C} \times B$ is such that a local inverse ϕ of ev_r^* near $\gamma(0)$ can be continued along γ upto but not including $\gamma(1)$. Using ϵ_j-wideness on X_j we conclude: For any j there is a parameter $t_j \in (0,1)$ such that $\phi \circ \gamma(t) \notin X_j$ for all $t \in (t_j, 1)$. This implies that $\pi_r^* \circ \phi \circ \gamma$ converges to a facet in Δ_∞ (i.e. every $\mathbb{C}^*$-stable open neighborhood of that facet contains a tail of the curve). However, if a curve in X is such that its π_r^* image tends to a facet in Δ_∞ the ev_r^* image of the curve tends to the boundary of $\mathbb{C} \times B$. That is to say, the ev_r^* image tends to be at an infinite distance from any point in $\mathbb{C} \times B$ with respect to the given metric. This is a consequence of theorem 3.21. (It is not hard to see, using that converge there is locally uniform and Cauchy's integral theorem, that this behaviour also holds on the π_r^* pre image of Δ). In particular the ev_r^* image of $\phi\circ\gamma$ should tend to the boundary of $\mathbb{C}\times B$. But this is just the curve γ, which tends to $\gamma(1)\in\mathbb{C}\times B$.

This contradiction shows that any local inverse of ev_r^* can be continued throughout $\mathbb{C}\times B$. Hence ev_r^* has a single valued holomorphic inverse on $\mathbb{C}\times B$. This shows that ev_r^* maps $X_r^*(p)$ globally biholomorphically onto $\mathbb{C}\times B$.□

Write $\nu/z=-d/a,\ d,a\in\mathbb{Z}_+,\ \gcd(d,a)=1$

Corollary 3.12 *Any local inverse of the multivalued mapping* $Rev(k):X\to\mathbb{B}$ *extends holomorphically to the d-fold covering of* $\mathbb{B}$*, and to no other covering of smaller degree.*

Proof: Let ϕ denote the inverse of ev_r^*. The map $\pi_r^*\circ\phi:\mathbb{C}\times B\to\mathbb{C}^n\backslash\Delta_\infty$ is globally holomorphic on $\mathbb{C}\times B$ and the lift of a local inverse of $Rev(k)$. Now by the relation

$$\mathrm{ev}_r^*(x\cdot y)=\nu x\cdot\mathrm{ev}_r^*(y),\ x\in\mathbb{C},\ y\in X_r^*(p)$$

and the fact that for a generic point $y\in X_r^*(p)$ we have

$$\pi_r^*(x_1\cdot y)=\pi_r^*(x_2\cdot y)\ \Leftrightarrow\ x_1-x_2\in\mathbb{Z}/z$$

we conclude that $\pi_r^*\circ\phi$ is invariant under the action of $t\in\mathbb{Z}/z$ iff t is a multiple of d/z. Hence by dividing out the action of $d\mathbb{Z}/z$ on $\mathbb{C}\times B$, the map $\pi_r^*\circ\phi$ descends to a globally holomorphic extension of a local inverse of $Rev(k)$ on the d-fold covering of $\mathbb{B}$. It is clear that the degree d is minimal in this sense. □

Corollary 3.13 *If all connected sub diagrams of* (M,p) *are either elliptic or parabolic, then the geometric realisation* $G(M,p)$ *of* $B(M,p)$ *has the following presentation:*

$$\begin{array}{lll}\langle r_1,\dots,r_n & | & r_i^{p_i}=e,\ i\in\{1,\dots,n\}\\ & & (r_i,r_j)^{m_{ij}}=(r_j,r_i)^{m_{ji}},\ 1\le i<j\le n\\ & & (r_1r_2\cdots r_n)^{ha/z}=e\rangle\end{array}$$

Proof: The biholomorphic equivalence of $X_r^*(p)$ and $\mathbb{C}\times B$ shows that

$$B(M,p)/N\cong G(M,p)$$

where N denotes the $\tilde\rho$ pre image of the kernel of

$$\mathrm{pr}:\widetilde{G}\to\rho^*(k,B(M))\cong G(M,p).$$

This kernel consists exactly of translations of $\mathbb{C}\times B$ in the first factor over an integral multiple of $1/z$. Relating both $\mathbb{C}$-actions on $X_r^*(p)$ and $\mathbb{C}\times B$ by the transformation formula for ev_r^* from the proof of corollary 3.12, we conclude that N is generated by $(g_1 g_2 \cdots g_n)^{ha/z}$. The corollary follows. $\square$

We conclude this section by formulating a Chevalley theorem for hyperbolic reflection groups.

A holomorphic function $f : \mathbb{C} \times B \to \mathbb{C}$ with the property

$$f(x + t\frac{\nu}{z}, b) = e^{2\pi i t} f(x,b), \text{ for all } t \in \mathbb{C} \text{ and } (x,b) \in \mathbb{C} \times B$$

can be considered as a global section in a line bundle $\mathcal{L}$ over B. The group $\widetilde{G}$ acts naturally on $\mathcal{L}$ and the kernel of the projection of $\widetilde{G}$ onto $G(M,p)$ acts trivially. Hence $\mathcal{L}$ is a $G(M,p)$-homogeneous bundle. Consider the graded algebra

$$A := \bigoplus_{n \geq 0} \Gamma(B, \mathcal{L}^{\otimes n})$$

and let A^G denote the sub algebra of $G(M,p)$-invariant elements.

Theorem 3.23 *The algebra A^G of invariant sections is isomorphic to a* poly-nomial *algebra* $\mathbb{C}[\phi_1, \ldots, \phi_n]$.

Proof: Let $\widehat{ev} : \mathbb{C}^n \backslash \Delta_\infty \to \mathbb{C} \times B$ be a lifting of $Rev(k)$. Let $\phi = (\phi_1, \ldots, \phi_n) : \mathbb{C} \times B \to \mathbb{C}^n \backslash \Delta_\infty$ be the inverse of $\widehat{ev}$. Clearly the coordinates $\phi_1, \ldots, \phi_n$ are algebraically independent over $\mathbb{C}$. Using homogeneity of the evaluation mapping one deduces that ϕ_j is a global invariant section in $\mathcal{L}^{\otimes (d_j/z)}$. Now let $f \in \Gamma(B, \mathcal{L}^{\otimes n})$ be an invariant section (as a function on $\mathbb{C} \times B$). The composition $f \circ \widehat{ev}$ is invariant under monodromy and weighted homogeneous of degree n. Hence this composition extends to a *polynomial* on $\mathbb{C}^n$. This implies that f is a polynomial in $\phi_1, \ldots, \phi_n$. $\square$

A well known result of Selberg [Se, lemma 8] implies that $G(M,p)$ has a normal subgroup Π of finite index that acts *freely* on the complex ball B. On the smooth variety $\Pi \backslash B$ one can introduce a line bundle $\mathcal{L}$ as above, homogeneous with respect to the finite group $G(M,p)/\Pi$ generated by reflections. Then one can prove a Chevalley like theorem on the invariant sections in the algebra generated by $\Gamma(\Pi \backslash B, \mathcal{L})$. This is similar to the result of Milnor in [N] on the complex disc (one dimensional hyperbolic space).

3.8 A proof of theorem 3.20

In this section we present a proof of theorem 3.20. Let $e_1, \ldots, e_n$ be a basis of $\mathcal{E}_S$ as in section 3.3. Denote the dual sections in $\mathcal{E}_S^*$ by e_j^*. Defining $H^*(e_i^*, e_j^*) := H_{ij}^*$ (as in definition 3.12) provides a hermitean structure on the subbundle of

$\mathcal{E}_S^*$ over the real valued multiplicity functions $K'_{\mathbb{R}}$. To prove theorem 3.20 it suffices to show that

$$H^*(Rev(k), Rev(k)) > 0$$

on X for hyperbolic k.

Now let $\iota : \mathbb{C}^2 \to V$ be an injective linear mapping such that $\iota(\mathbb{C}^2 \setminus \{0\})$ intersects every reflection plane only in sub regular points. (In particular, the ι image is not contained in any reflection plane.) By Chevalley projection we get a weighted homogeneous mapping $\iota_P := P \circ \iota$ into $\mathbb{C}^n$ such that its image intersects $\Delta \setminus \{0\}$ only in subregular points. Let $a_1, \ldots, a_m$ be the lines in $\mathbb{C}^2$ which ι_P maps into Δ. Define a real valued function ϕ on $K'_{\mathbb{R}} \times \mathbb{C}^2 \setminus \{a_1, \ldots, a_m\}$ by:

$$\phi(k, x) := H^*(Rev(k, \iota_P(x)), Rev(k, \iota_P(x)))$$

Note that by monodromy invariance of H^* this defines a *single valued* continuous function. By the characterization in theorem 3.13 we conclude that ϕ extends to a continuous function (also called ϕ) on $K'_{\mathbb{R}} \times \mathbb{C}^2$. Also note that $\phi(k, \cdot)$ is homogeneous (of degree $\nu(k)$) for each k.

We now investigate if this ϕ can take on negative values. First observe that $\phi(k, x) > 0$ if $\nu(k) \geq 0$. Define N by:

$$N := \{(k, x) \in K'_{\mathbb{R}} \times \mathbb{C}^2 \mid \phi(k, x) \leq 0\}$$

(The set where ϕ takes on non positive values.) Then N is closed. Because N is invariant under scalar multiplication in the second factor and $\mathbb{P}(\mathbb{C}^2)$ is compact, we conclude that the projection N_K of N on $K'_{\mathbb{R}}$ along $\mathbb{C}^2$ is also closed.

Now suppose $k \in \partial N_K$. Then $\phi(k, \cdot) \geq 0$ and $\phi(k, x_o) = 0$ for some $x_o \in \mathbb{C}^2$. Suppose that $\nu(k) > 1 - m_2$. By a previous remark we necessarily have $\nu(k) < 0$. Because $Rev(k)$ is locally biholomorphic on X and $\iota_P(\mathbb{C}^2 \setminus \{0\})$ is not contained in a single (weighted) $\mathbb{C}^*$-orbit, we conclude that $\phi(k, x) = 0$ implies that $x \in a_1 \cup \ldots \cup a_m$. Hence $\phi(k, \cdot)$ vanishes along some line, a_1 say.

By theorem 3.13 we know that at a non zero point x_o in a_1 we can locally split $Rev_P := Rev(k, \iota_P(\cdot))$ in a singular and a holomorphic part:

$$Rev_P = Rev_s + Rev_h$$

In particular

1. $H^*(Rev_s, Rev_h) = 0$
2. Rev_h is holomorphic in a neighborhood of $x_o \in a_1$.
3. If $\iota_P(x_o)$ lies on a type j reflection plane, then Rev_s is a special eigenvector of $\rho^*(k, g_j)$ on $\mathcal{E}_S^*(k)$ (if non zero).
4. $\lim_{x \to x_o} Rev_s(x) = 0$

It is a consequence of property 3 that $H^*(Rev_s, Rev_s) \leq 0$. (H^* is negative on all special eigenvectors of the generating reflections.) Near x_o this yields:

$$0 \leq \phi(k, \cdot) = H^*(Rev_s, Rev_s) + H^*(Rev_h, Rev_h) \leq$$

$$\leq H^*(Rev_h, Rev_h)$$

By the maximum principle (section 1.1) we conclude that

$$H^*(Rev_h, Rev_h) > 0$$

on a neighborhood of x_o. This is in contradiction with the fact that $Rev(k, x_o) = 0$. We conclude that if $k \in \partial N_K$ then $\nu(k) \leq 1 - m_2$. Now the subset of $K'_{\mathbb{R}}$ for which $\nu(k) > 1 - m_2$ is connected and not contained in N_K. We conclude that it is *disjoint* from N_K. This shows that $\phi(k, x) > 0$ if $\nu(k) > 1 - m_2$. In particular we conclude that on the ι_P image of $\mathbb{C}^2$, restricted evaluation maps into $\mathbb{B}$. Theorem 3.20 now follows by the remark that by varying the map ι, the images of ι_P cover X.□

3.9 Literature

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Chapter 4

Root systems and varieties

4.1 Introduction

In chapter 3 a hypergeometric system related to a normalized root system R was introduced. It is a local system or, in Deligne's terminology, a function of Nilsson class on the complement of the discriminant of R. This system depends on a multiplicity parameter k and some conditions on this parameter were introduced that will assure that the hypergeometric system has a discrete monodromy group. One of these conditions is that all proper parabolic root subsystems of R with the restricted multiplicity parameter should have a non-negative exponent.

Studying [DM] shows that this condition is certainly not necessary in general, though it marks an important border in the theory. One should compare this with the theory of *real* hyperbolic reflection groups that act discretely and with cofinite volume on real hyperbolic space. There is a considerable difference in effort needed to classify such groups with at most parabolic subgroups (as in [H]) and the general case [V].

The presented work is mainly concerned with constructions of varieties and describing their properties. First a "Cremona" variety of a restricted Coxeter arrangement is introduced. Then we generalize the appearance of Geometric Invariant Theory for the root system A_n, present in the work of Deligne and Mostow [DM], to arbitrary root systems. This will result in a better understanding of hypergeometric systems associated to root systems with a proper root subsystem of hyperbolic type.

Unfortunately, there remain some questions in the "invariant theory" for arbitrary root systems. Therefore the final main results are still conjectural in nature.

The main conjecture of this chapter can be formulated as follows. Let k be a multiplicity parameter on an irreducible root system R such that

$$k_\alpha = \frac{1}{2} - \frac{1}{p_\alpha}$$

for $p_\alpha \in \mathbb{N}_{\geq 2}$ and $\alpha \in R$.

Conjecture 4.1 *Suppose that $\nu(R,k) \in (1-m_2, 0)$ where m_2 denotes the second smallest exponent of R, i.e. $\nu(R,k)$ is of hyperbolic type. Suppose moreover that for any irreducible parabolic root subsystem $R' \subset R$ of rank rk$(R)-1$ such that $\nu(R',k) < 0$ the following integrality condition holds:*

$$-\varepsilon_{R'}/\nu(R',k) \in \mathbb{N}_{\geq 1}.$$

Here $\varepsilon_{R'} \in \{1,2\}$ and it equals 2 exactly if $W(R)$ contains an element w such that $\pm w$ is a reflection fixing R'. Then the monodromy group of the hypergeometric system $\mathcal{E}_S(k)$ (cf. section 3.3) is discrete.

This results in the tables of chapter 5.

4.2 Coxeter arrangements

In this section we introduce the notion of a *Coxeter arrangement*. Let R be a normalized irreducible root system of full rank in an n-dimensional Euclidean space $(E, (\cdot,\cdot))$. Define $V := \mathbb{C}\otimes E$ and extend $(\cdot,\cdot)$ bilinearly to V. For a subset $U \subseteq E$ we define $V_U := \mathrm{span}_{\mathbb{C}}(U)$ and $V^U := V_U^\perp$. In particular $V = V_U \oplus V^U$. A root system $R' \subseteq R$ is called *parabolic* if $R' = V_{R'} \cap R$. For $R' \subseteq R$ a parabolic root subsystem we define:

$$\mathcal{R}(R',R) := \{S \subseteq R \mid R' \subseteq S \text{ and } S \text{ is parabolic}\}$$

$$\mathcal{S}(R',R) := \{S \in \mathcal{R}(R',R) \mid \mathrm{rk}(S) = \mathrm{rk}(R') + 1\}$$

If R' is irreducible we define

$$\mathcal{R}_o(R',R) := \{S \in \mathcal{R}(R',R) \mid S \text{ is irreducible}\}$$

$$\mathcal{S}_o(R',R) := \{S \in \mathcal{S}(R',R) \mid S \text{ is irreducible}\}$$

$$N(R',R) := \#\mathcal{S}_o(R',R)$$

Example: The root system of type E_8 contains D_5 as an irreducible parabolic root subsystem. In this case $\mathcal{S}_o(D_5, E_8)$ contains four systems of type E_6 and three of type D_6. Therefore $N(D_5, E_8)$ is equal to seven.

Fix a root subsystem $R' \in \mathcal{R}_o(\emptyset, R)$ such that $R' \neq \emptyset$. For every $\alpha \in R \backslash R'$ the linear space $\alpha^\perp \cap V^{R'}$ is of codimension one in $V^{R'}$. Two such roots can have the same orthoplement in $V^{R'}$ even if they are linearly independent. Take $\alpha \in R \backslash R'$ and consider the set

$$\{\beta \in R \mid \beta^\perp \supseteq \alpha^\perp \cap V^{R'}\}.$$

This is a parabolic root system of rank $\mathrm{rk}(R') + 1$ containing R'. It is either irreducible or of the form $R' \cup \{-\alpha, \alpha\}$. All roots α for which this system is reducible form a subset $(R')^{\perp\!\!\perp}$ of $(R')^\perp$. In fact $(R')^{\perp\!\!\perp}$ is a (not necessarily parabolic) root subsystem of R. For example if $R = B_n$ and $R' = B_m$ for some $m \leq n - 4$ then $(R')^{\perp\!\!\perp}$ is of type D_{n-m}.

The hyperplanes $\alpha^\perp \cap V^{R'}$ for $\alpha \in R \backslash R'$ are exactly indexed by $\mathcal{S}(R', R)$.

Definition 4.1 *The space $V^{R'}$ stratified by the intersection structure of all hyperplanes V^S, $S \in \mathcal{S}(R', R)$ is called a* restricted Coxeter arrangement *[OT].*

Let us study the intersection structure of all hyperplanes. Take inclusion as a partial ordering on $\mathcal{R}(R', R)$.

Lemma 4.1 *In $\mathcal{R}(R', R)$ any two elements S, S' have a least upper bound $S \vee S'$ and a greatest lower bound $S \wedge S'$.*

Proof: Let $S, S' \in \mathcal{R}(R', R)$ then clearly $(V_S + V_{S'}) \cap R$ is a parabolic root subsystem containing both S and S'. Moreover any upper bound for S and S' must contain $(V_S + V_{S'}) \cap R$ because it is parabolic. Because $\mathcal{R}(R', R)$ is finite S and S' will also have a greatest lower bound.□

Lemma 4.2 *If $S \in \mathcal{R}(R', R)$ then $T \mapsto T \vee S$ defines a map*

$$\mathcal{S}(R', R) \backslash \mathcal{S}(R', S) \to \mathcal{S}(S, R').$$

Moreover, this map is onto but not necessarily injective.

Proof: If $T \in \mathcal{S}(R', R) \backslash \mathcal{S}(R', S)$ then indeed $\mathrm{rk}(T \vee S) = \mathrm{rk}(S) + 1$. If $S' \in \mathcal{S}(S, R)$ and $\alpha \in S' \backslash S$ then $T := (V_{R'} + \mathbb{C}\alpha) \cap R$ is an element of $\mathcal{S}(R', R) \backslash \mathcal{S}(R', S)$ and $S' = T \vee S$. Take as an example $R = B_4$, $R' = A_1$, $S = B_2$. Then one checks that this map is not injective.□

Corollary 4.1 *Any element of $\mathcal{R}(R', R)$ is the least upper bound of a subset of $\mathcal{S}(R', R)$.*

Proof: Induction on the rank. If $S' \in \mathcal{R}(R', R)$, $S' \neq R'$ then S' contains a parabolic root subsystem S of corank one in S'. By induction and lemma 4.2 we find a subset of $\mathcal{S}(R', R)$ for which S' is the least upper bound. □

Corollary 4.2 *The set of all intersections of hyperplanes V^S in $V^{R'}$, $S \in \mathcal{S}(R', R)$, partially ordered by reversed inclusion is isomorphic to $\mathcal{R}(R', R)$ (as partially ordered sets).*

Proof: This follows from corollary 4.1 and the fact that $V^S \cap V^{S'} = V^{S \vee S'}$. □

The hyperplanes V^S, $S \in \mathcal{S}_o(R', R)$ play a special role in the next section. To prove some properties of the intersection structure of these hyperplanes we need the following lemma.

Lemma 4.3 *If a root subsystem $S \subseteq R$ is irreducible and $A \subset V_S$ is a proper linear subspace, then $S \subset V_{S \backslash A}$.*

Proof: If $\alpha \in A \cap S$ then $S \backslash A$ is invariant under reflection in α. Moreover if α is not perpendicular to $S \backslash A$ then it is contained in $V_{S \backslash A}$. Hence $V_S = V_{S \backslash A} \oplus V'$ with $V' = \mathrm{Span}_{\mathbb{C}}\{\alpha \in S \mid \alpha \perp S \backslash A\}$. By irreducibility of S we have $V' = \{0\}$. □

Lemma 4.4 *Lemma 4.1, lemma 4.2, corollary 4.1 and corollary 4.2 still hold if one replaces $\mathcal{R}$ by $\mathcal{R}_o$ and $\mathcal{S}$ by $\mathcal{S}_o$.*

Proof: If $S, S' \in \mathcal{R}_o(R', R)$ then $S \cap S' \neq \emptyset$ and hence $S \vee S'$ is irreducible proving lemma 4.1. If $S \in \mathcal{R}_o(R', R)$ and $S' \in \mathcal{S}_o(S, R)$ then $S' \backslash S$ is not perpendicular to R' by lemma 4.3 and the fact that $R' \subset S'$. Hence for $\alpha \in S' \backslash S$ the parabolic system $(V_{R'} + \mathbb{C}\alpha) \cap R$ is irreducible proving lemma 4.2. Corollary 4.1 follows from the remark that an irreducible parabolic root system contains an irreducible parabolic root subsystem of corank one. Corollary 4.2 is then clear. □

Remark 4.1 *If $S, S' \in \mathcal{R}_o(R', R)$ then the greatest lower bound of S and S' in* $\mathcal{R}(R', R)$ need not be irreducible. *The irreducible component containing R' is the greatest lower bound in $\mathcal{R}_o(R', R)$. So the exact meaning of $S \wedge S'$ depends on the context.*

For $S \in \mathcal{R}(R', R)$ we denote the complement of all $V^{S'}$ in V^S, $S' \in \mathcal{S}(S, R)$, by $\mathcal{H}^S(R)$ or $\mathcal{H}^S$. Likewise for $S \in \mathcal{R}_o(R', R)$ we denote the complement of all $V^{S'}$ in V^S, $S' \in \mathcal{S}_o(S, R)$ by $\mathcal{H}_o^S(R)$ or $\mathcal{H}_o^S$.

Let the subgroup $W(R', R)$ of $W(R)$ be defined as the set of elements $w \in W(R)$ such that $w_{|V_{R'}} = \pm \mathrm{id}_{V_{R'}}$.

Lemma 4.5 *The group $W(R', R)$ is generated by reflections keeping R' pointwise fixed and at most one element $w_- \in W(R)$ such that $w_-(v) = -v$ for all $v \in V_{R'}$.*

Let $w_- \in W(R',R)$ be any element such that $w_-(v) = -v$ for all $v \in V_{R'}$ (if such an element exists). It is well known that the group of elements fixing R' pointwise is generated by reflections. Take $w \in W(R',R)$ and suppose that $w(v) = -v$ for all $v \in V_{R'}$. Then ww_- fixes R' pointwise. Hence $w = ww_- \cdot w_-^{-1}$ so w is a product of reflections fixing R' and w_-^{-1}. This proves the lemma. $\square$

Note that $V^{R'}$ is stable under $W(R',R)$. Therefore the following definition of a $W(R',R)$-action on $V^{R'}$ might be unexpected.

Definition 4.2 *Define a $W(R',R)$-action on $V^{R'}$ by*

$$w.v = \begin{cases} w(v) & \text{if } w \text{ fixes } R' \\ -w(v) & \text{otherwise} \end{cases}$$

for any $w \in W(R',R)$ and $v \in V^{R'}$.

In section 4.5 it will become clear why this is a natural action for our purposes.

Lemma 4.6 *If R is not of type D_n (n odd) nor of type E_6 then for any $w \in W(R',R)$ there exists a $\tilde{w} \in W(R)$ fixing R' such that $w.v = \tilde{w}(v)$ for all $v \in V^{R'}$. In this case $W(R',R)$ acts freely on $\mathcal{H}^{R'}$.*

Proof: If R is not of type D_n (n odd) or E_6 then either $w(v) = v$ for all $w \in W(R',R)$ and $v \in V_{R'}$ or $-1 \in W(R',R)$. In the latter case one can take $\tilde{w} = -w$. Now the group of elements fixing R' acts freely on $\mathcal{H}^{R'}$. $\square$

Lemma 4.7 *If R is of type E_6 and $w \in W(R',E_6)$, $v \in \mathcal{H}^{R'}$ are such that $w.v = v$ then the fixed points of w on $V^{R'}$ (with respect to the dot action) form a linear space of codimension at least two.*

Proof: If $w \in W(R',E_6)$ fixes a linear subspace of $V^{R'}$ of codimension one then either w or $-w$ is a reflection. In the first case w has no fixed points on $\mathcal{H}^{R'}$ by definition. If $-w$ would be a reflection then w fixes a one dimensional facet of E_6 in $V^{R'}$. Hence -1 would be an element of the stabilizer of this facet. Now such a stabilizer is the reflection group of a root system of one of the following types: D_5, $A_1 \times A_4$, $A_2 \times A_2 \times A_1$ or A_5. In particular -1 is not an element of such a stabilizer and hence $-w$ can not be a reflection. $\square$

Corollary 4.3 *If $R' \subset E_6$ is of rank four then $W(R',R)$ acts freely on $\mathcal{H}^{R'}(E_6)$.*

Proof: A non trivial linear subspace in $V^{R'}$ has corank one. $\square$

Remark 4.2 *Unfortunately an analogue of lemma 4.7 does not hold if R is of type D_n (n odd). Consider R' of type A_m, $m \leq n-2$. Then the longest element* *in $D_{n-1} \supset A_m$ fixes a subspace of $\mathcal{H}^{A_m}$ of codimension one that is not contained in any of the spaces $\mathcal{H}^S$, $S \in \mathcal{S}(A_m, D_n)$.*

We exclude the possibility for R to be of type D_n (n odd) in the rest of this chapter. *(This is really not a bad restriction because these cases are essentially covered by type B_n).*

4.3 The Cremona cone

In this section R has rank n and is not of type D_n (n odd) and R' is again a fixed non-empty parabolic irreducible root subsystem of R. Certain varieties associated to Coxeter arrangements are constructed. We use hypergeometric functions associated to R to study ramified coverings of such varieties modulo a $W(R', R)$ action.

For all $S \in \mathcal{S}_o(R', R)$ let $\beta_S \in E$ be a vector in $V^{R'} \cap V_S$ normalized by $(\beta_S, \beta_S) = 2$. All vectors β_S together span $V^{R'}$ because R is the least upper bound for $\mathcal{S}_o(R', R)$. The linear form $(\cdot, \beta_S)$ on $V^{R'}$ is denoted as β_S^*. Let y_S, $S \in \mathcal{S}_o(R', R)$ be coordinates on $\mathbb{C}^{N(R',R)}$. Define a map $\gamma_{R',R} : \mathcal{H}_o^{R'} \to \mathbb{C}^{N(R',R)}$ by

$$\gamma = \gamma_{R',R} : v \mapsto (\frac{1}{(v, \beta_S)})_{S \in \mathcal{S}_o(R',R)}.$$

Note that γ is a smooth injective homogeneous map of degree -1. Define $\Gamma^o \subset V^{R'} \times \mathbb{C}^{N(R',R)}$ by

$$\Gamma^o := \{(v, y) \mid v \in \mathcal{H}_o^{R'} \text{ and } \gamma(v) = \lambda y \text{ for some } \lambda \in \mathbb{C}^*\}.$$

Then Γ^o is $\mathbb{C}^*$-invariant in both factors separately. Let $\Gamma \subset V^{R'} \times \mathbb{C}^{N(R',R)}$ be the topological closure of Γ^o. For a set $Y \subseteq V^{R'}$ we define a set

$$\Gamma(Y) := \{y \in \mathbb{C}^{N(R',R)} \mid (v, y) \in \Gamma \text{ for some } v \in Y\}.$$

Lemma 4.8 *For every set $Y \subseteq V^{R'}$ the set $\Gamma(Y)$ is $\mathbb{C}$-invariant. If moreover $\mathbb{C}Y$ is closed then $\Gamma(Y \backslash \{0\})$ is also closed. In particular $\Gamma(\{y\}) = \Gamma(\{\mathbb{C}^* y\})$ is closed for all $y \in V^{R'}$.*

Proof: Left to the reader. □

Now $\Gamma(\{0\})$ is exactly the closure of $\gamma(\mathcal{H}_o^{R'})$ in $\mathbb{C}^{N(R',R)}$. We denote this closure by $\mathrm{Cone}(R', R)$ and call it the *Cremona cone* of the arrangement of R' in R.

Remark 4.3 *In this way Γ can be viewed as a birational map between $\mathbb{P}(V^{R'})$ and $\mathbb{P}(Cone(R', R))$.*

Example: If $R = A_n$ and $R' = A_m$ for some $1 \leq m < n$ then $\mathcal{S}_o(A_m, A_n)$ contains exactly $n - m$ root systems of type A_{m+1}. In this case $\mathrm{Cone}(A_m, A_n)$ equals $\mathbb{C}^{n-m}$. More generally $\mathrm{Cone}(R', R) = \mathbb{C}^{N(R',R)}$ exactly if $N(R', R) = \mathrm{rk}(R) - \mathrm{rk}(R')$.

Note that $\Gamma(\mathcal{H}_o^R) = \Gamma(\{0\})$ equals $\mathrm{Cone}(R', R)$ (by definition). More generally we have the following.

Theorem 4.1 *For $S \in \mathcal{R}_o(R', R) \backslash \{R'\}$ define $\Gamma_S := \Gamma(\mathcal{H}_o^S)$ and take $\Gamma_{R'} := \{0\}$. Take $S, S' \in \mathcal{R}_o(R', R)$ such that $S \neq R'$. Then for any $v \in \mathcal{H}_o^S$ we have $\Gamma(\{v\}) = \Gamma_S$ and as a variety Γ_S is isomorphic to Cone(R', S). The intersection $\Gamma_S \cap \Gamma_{S'}$ exactly equals $\Gamma_{S \wedge S'}$ (irreducible greatest lower bound).*

Proof: Take $v \in \mathcal{H}_o^S$. Let $U_1 \subset \mathcal{H}_o^S$ be a neighborhood of v such that also $\overline{U_1} \subset \mathcal{H}_o^S$. Let $U_2 \subset V_S \cap V^{R'}$ be a neighborhood of 0 such that $\overline{U_2}$ is compact. Let $\varepsilon \in \mathbb{C}^*$ be small and $u_1 \in U_1$, $u_2 \in U_2$ such that $(u_2, \beta_T) \neq 0$ for all $T \in \mathcal{R}_o(R', S)$, i.e. $u_2 \in \mathcal{H}_o^{R'}(S)$. Then $(u_1 + \varepsilon u_2, \gamma(\varepsilon^{-1} u_1 + u_2)) \in \Gamma^o$. The coordinate y_T of $\gamma(\varepsilon^{-1} u_1 + u_2)$, $T \in \mathcal{S}_o(R', R)$, is given by:

$$y_T = \begin{cases} \dfrac{1}{(u_2, \beta_T)} & \text{if } T \in \mathcal{S}_o(R', S) \\ \dfrac{\varepsilon}{(u_1, \beta_T) + \varepsilon(u_2, \beta_T)} & \text{otherwise} \end{cases}$$

Now let ε tend to 0. Then the coordinates y_T for $T \notin \mathcal{S}_o(R', S)$ tend to 0 uniformly in u_1, u_2 while those for $T \in \mathcal{S}_o(R', S)$ are uniformly bounded below. This shows that if $y \in \Gamma(\{v\})$ then its coordinate y_T can be non-zero only if $T \in \mathcal{S}_o(R', S)$. On the other hand if $y \in \mathrm{Cone}(R', R)$ and all coordinates y_T are zero for $T \in \mathcal{S}_o(R', R) \backslash \mathcal{S}_o(R', S)$ then it is not hard to check that y is contained in the closure of

$$\{(\frac{1}{(u_2, \beta_T)})_{T \in \mathcal{S}_o(R', S)} \mid u_2 \in \mathcal{H}_o^{R'}(S)\}$$

embedded in $\mathbb{C}^{N(R',R)}$. So Γ_S is isomorphic to $\mathrm{Cone}(R', S)$ and $\Gamma_S \cap \Gamma_{S'}$ equals $\Gamma_{S \wedge S'}$. $\square$

For $S \in \mathcal{R}_o(R', R)$ we define $\mathcal{G}_S$ as the complement of all $\Gamma_{S'}$ in Γ_S, $S' \in \mathcal{R}_o(R', S) \backslash \{S\}$. It is the $\gamma_{R',S}$-image of $\mathcal{H}_o^{R'}(S)$ embedded in $\mathrm{Cone}(R', R)$.

Theorem 4.2 *Suppose $y \in \mathcal{G}_S$ for $S \in \mathcal{R}_o(R', R)$ and let $m = rk(S) - rk(R')$, i.e. $m = \dim(\Gamma_S)$. Then y has a neighborhood in Cone(R', R) which is isomorphic to a product*

$$\Delta^m \times (\Delta^{N(S,R)} \cap \mathit{Cone}(S, R))$$

where $\Delta \subset \mathbb{C}$ denotes the unit disc. In this neighborhood $\mathcal{G}_S$ corresponds to $\Delta^m \times \{0\}$. In particular $\mathcal{G}_S$ is smoothly embedded in Cone(R', R) if and only if $N(S, R)$ equals $rk(R) - rk(S)$.

Proof: Fix $v_1 \in \mathcal{H}_o^S$ and $v_2 \in \mathcal{H}_o^{R'}(S)$. Let $U_2 \subset \mathcal{H}_o^{R'}(S)$ be a neighborhood of v_2 and $\delta > 0$ such that

$$|(u, \beta_T)| \leq \delta^{-1}|(v_1, \beta_T)|$$

for all $u \in U$ and $T \in \mathcal{S}_o(R', R)\backslash\mathcal{S}_o(R', S)$.

The following formulas are inspired by those for y_T above. Take $(u, x) \in U \times (\delta\Delta^{N(S,R)} \cap \text{Cone}(S, R))$ and define the point $y(u, x) \in \mathbb{C}^{N(R',R)}$ by its coordinates:

$$y_T(u, x) := \begin{cases} \dfrac{1}{(u, \beta_T)} & \text{if } T \in \mathcal{S}_o(R', S) \\ \dfrac{x_{T\vee S}}{(v_1, \beta_T) + x_{T\vee S}(u, \beta_T)} & \text{otherwise} \end{cases}, \ T \in \mathcal{S}_o(R', R)$$

Then one can check that $y(u, x) \in \text{Cone}(R', R)$. Indeed if we take $x \in \text{Cone}(S, R)$ given by

$$x_{T\vee S} := \frac{(v_1, \beta_T)\varepsilon}{(u_1, \beta_T)}$$

for $T \in \mathcal{S}_o(R', R)\backslash\mathcal{S}_o(R', S)$ and some $u_1 \in \mathcal{H}_o^S$ (this x is well defined) then $y(u, x)$ is just $\gamma(\varepsilon^{-1}u_1 + u)$. The map $(u, x) \mapsto y(u, x)$ is biholomorphic on $U \times (\delta\Delta^{N(S,R)} \cap \text{Cone}(S, R))$. Moreover, $y(u, x) \in \Gamma(\mathcal{H}_o^S)$ precisely if $x = 0$.

Recall that $N(S, R) = \text{rk}(R) - \text{rk}(S)$ implies $\text{Cone}(S, R) = \mathbb{C}^{N(S,R)}$. If $N(S, R)$ is greater however then 0 is a singular point of $\text{Cone}(S, R)$. The theorem follows.$\square$

If $w \in W(R', R)$ then $w.\beta_S = \sigma(w, S)\beta_{w.S}$ for some $\sigma(w, S) \in \{-1, 1\}$. Define a $W(R', R)$-action on $\mathbb{C}^{N(R',R)}$ by:

$$W(R', R) \ni w^{-1} : (y_S)_{S\in\mathcal{S}_o(R',R)} \mapsto (\sigma(w, S)y_{w.S})_{S\in\mathcal{S}_o(R',R)}$$

So $W(R', R)$ acts by sign changes and permutations of the coordinates. The important property of this action is that it makes γ a $W(R', R)$-equivariant map. Hence Γ is stable under the diagonal $W(R', R)$-action on $V^{R'} \times \mathbb{C}^{N(R',R)}$. In particular the action on $\mathbb{C}^{N(R',R)}$ restricts to an action on $\text{Cone}(R', R)$. In all cases except possibly if R is of type E_6 this action will be free on $\gamma(\mathcal{H}^{R'})$ (lemma 4.6) and we call this set the *regular part* of $\text{Cone}(R', R)$.

Recall that $\mathcal{G}_S$ has codimension one in $\text{Cone}(R', R)$ precisely if $\text{rk}(S) = n - 1$. In this case $\mathcal{G}_S$ is smoothly embedded.

Theorem 4.3 *Suppose $S \in \mathcal{R}_o(R', R)$ has rank $n-1$. An element $w \in W(R', R)$ acting non trivially fixes Γ_S pointwise if and only if w acts as a reflection on $V^{R'}$ fixing V_S (by the dot action). In particular it is an involution on Cone(R', R).*

Proof: By invariance of Γ and theorem 4.1 we conclude that V^S and hence $V_S \cap V^{R'}$ are stable under w. Now $\mathcal{G}_S$ corresponds to $\mathcal{H}^{R'}(S)$ by the w-equivariant map $\gamma_{R',S}$. In particular $\mathcal{G}_S$ is pointwise fixed if and only if $V_S \cap V^{R'}$ is pointwise fixed. Because w is non-trivial it must act as a reflection.$\square$

Next we study the roots in $(R')^{\perp\!\!\perp}$ and the fixed point sets on $\mathrm{Cone}(R', R)$ of the corresponding reflections. If $\alpha \in (R')^{\perp\!\!\perp}$ let Γ_α be the set of fixed points on $\mathrm{Cone}(R', R)$ of the reflection s_α with root α.

Lemma 4.9 *Suppose $\alpha \in (R')^{\perp\!\!\perp}$ and $S \in \mathcal{R}_o(R', R)$. The set Γ_α intersects $\mathcal{G}_S$ if and only if S is stable under the reflection s_α.*

Proof: Clearly $W(R', R)$ permutes the sets $\mathcal{G}_{S'}$, $S' \in \mathcal{R}_o(R', R)$. In particular if s_α has a fixed point on $\mathcal{G}_S$ then $\mathcal{G}_S$ is stable under s_α. By invariance of Γ this implies that $\mathcal{H}_o^S$ and hence S is s_α-stable. If $\alpha \perp S$ then Γ_S is pointwise fixed by s_α. If $\alpha \in S$ then s_α has a fixed point on $\mathcal{H}_o^{R'}(S)$ and hence on $\mathcal{G}_S$ (essentially the $\gamma_{R',S}$-image of the former set). $\square$

For an arbitrary collection of such reflections the following holds.

Theorem 4.4 *Let $A \subseteq (R')^{\perp\!\!\perp}$ and $S \in \mathcal{R}_o(R', R)$. The reflections s_α, $\alpha \in A$ have a common fixed point on $\mathcal{G}_S$ if and only if the following conditions are satisfied.*

1. *The root system S is stable under every reflection s_α, $\alpha \in A$.*
2. *The root system R' is an orthogonal component of the smallest element in $\mathcal{R}(R', R)$ containing $A \cap S$.*

Proof: Condition one states that every s_α has a fixed point on $\mathcal{G}_S$ by the previous lemma. Let $T \in \mathcal{R}(R', R)$ be the smallest element containing $A \cap S$. Let T_o be the irreducible component of T containing R'. Clearly $T_o \in \mathcal{R}_o(R', S)$. Then the common fixed points of s_α, $\alpha \in A \cap S$ on $V^{R'}$ are contained in V^{T_o}. Moreover $\mathcal{H}_o^{R'}(S)$ contains common fixed points if and only if $T_o = R'$. Now fixed points on $\mathcal{G}_S$ correspond to fixed points on $\mathcal{H}_o^{R'}(S)$. The theorem follows. $\square$

In the theory of groups generated by reflections of some vector space it is well known that the stabilizer of any point is again generated by reflections. This fails in general for the action of $W(R', R)$ on $\mathrm{Cone}(R', R)$.

Theorem 4.5 *Suppose $S \in \mathcal{R}_o(R', R)$ and let $y \in \mathcal{G}_S$. Suppose the pair (R', R) is not any of the following: (A_1, A_p) with $p \geq 3$ odd, (A_p, D_q) with $p \leq q - 2$, (A_j, E_6) with $j \in \{1, 2, 3\}$. Then the stabilizer of y in $W(R', R)$ is the direct product of $W(S, R)$ and the subgroup of $W(S)$ generated by all reflections fixing R' and y.*

Proof: Let $w \in W(R', R)$ stabilize $y \in \mathcal{G}_S$. Then S is w-stable. Let $\varepsilon \in \{-1, 1\}$ be such that $w(v) = \varepsilon v$ for all $v \in V_{R'}$. Then $\varepsilon w(v) = v$ for all $v \in V_{R'}$ and some non-zero $v \in \mathcal{H}_o^{R'}(S) \cap E$. Let C be a chamber of S in $V_S \cap E$ such that C intersects the fixed points of εw in a facet of highest possible dimension. Because $\varepsilon w(C)$ is again a chamber and $\varepsilon w(C) \cap C \neq \emptyset$ there is a $g \in W(S)$ such that $\varepsilon g w(C) = C$. Moreover every fixed point of εw on V_S is fixed by g. In particular g is a product of reflections in $W(S)$ fixing R' and y.

The transformation $\varepsilon g w$ induces a diagram automorphism of S. If S admits no non-trivial diagram automorphisms then $gw \in W(S,R)$.

Remains to consider the cases where S is of type A_p, D_q or E_6 with a non-trivial diagram automorphism. In these cases the automorphism is an involution and the roots that are fixed form a root subsystem of type $A_1^{\lceil p/2 \rceil}$, D_{q-1} and D_4 respectively. This restricts the possibilities to (A_1, A_p), (A_p, D_q), (D_p, D_q), $(A_{1,2,3}, E_6)$ and (D_4, E_6). The condition that the involution should have fixed points in $\mathcal{H}_o^{R'}(S)$ and some explicit computations yield the list stated in the theorem. □

In the remainder of this section we will always assume that the pair (R', R) *is none of those listed in theorem 4.5. In particular this implies that* $W(R', R)$ *acts freely on* $\mathcal{H}^{R'}$ *as the only possible exceptions would be* (A_j, E_6), $j \in \{1,2,3\}$.

To study the structure of $\mathrm{Cone}(R', R)$ modulo the $W(R', R)$-action we introduce a function of Nilsson class on the regular part $\gamma(\mathcal{H}^{R'})$ of $\mathrm{Cone}(R', R)$ related to the hypergeometric function of the root system R.

Without loss of generality we can assume that R' is generated by the $n-m$ simple roots $\alpha_1, \ldots, \alpha_{n-m} \in R$ for some $m \geq 1$. Let v be a regular point in E^+ with orbit $W(R)v$. Then the hypergeometric system $\mathcal{E}_{W(R)v}(k)$ has a m-dimensional subspace of vectors kept fixed by the reflections $\rho(k, g_1), \ldots, \rho(k, g_{n-m})$. Moreover, any germ component in $\mathcal{E}_v(k)$ of such a fixed vector will extend holomorphically over any point in the space $\mathcal{H}^{R'} \subset V$. Let $x \in \mathcal{H}^{R'} \cap E$. By restriction we get a m-dimensional vectorspace $\mathcal{C}_x(k)$ of germs of multivalued functions on $\mathcal{H}^{R'}$ at the point x. Recall that for a multiplicity parameter k on R the *exponent* of R is defined as

$$\nu(R,k) := 1 - \frac{1}{n} \sum_{\alpha \in R} k_\alpha \in \mathbb{Z}[k_\alpha].$$

We will need the following remarkable equality between exponents of root systems which plays a crucial role in the sequel.

Theorem 4.6 *For* any *irreducible root system* R *and* any *parabolic irreducible root subsystem* R' *the following equality holds:*

$$\sum_{S \in \mathcal{S}_o(R', R)} (\nu(S,k) - \nu(R',k)) = \nu(R,k) - \nu(R',k)$$

Proof: Unfortunately the only proof I know at the moment is by an elaborate case by case verification using tables of the positive roots for all root systems R. □

To obtain a Nilsson class function on $\gamma(\mathcal{H}^{R'})$ we want to use the map γ to push forward the system $\mathcal{C}_x(k)$ on $\mathcal{H}^{R'}$. However, it turns out to be more convenient to push forward a slightly altered system on $\mathcal{H}^{R'}$ in order to obtain nice local properties on $\gamma(\mathcal{H}^{R'})$. We obtain this altered system $\mathcal{C}_x^{alt}(k)$ by tensoring $\mathcal{C}_x(k)$ with the one dimensional space spanned by a germ of the multivalued function

$$\prod_{S\in\mathcal{S}_o(R',R)} (\beta_S^*)^{\nu(R',k)-\nu(S,k)}$$

at $x \in \mathcal{H}^{R'}$. The following lemma states some important properties of this system.

Lemma 4.10 *1. Any germ in $\mathcal{C}_x^{alt}(k)$ is homogeneous of degree $\nu(R',k)$.*

2. For any $w \in W(R',R)$ there is a canonical isomorphism between the vector spaces $\mathcal{C}_x^{alt}(k)$ and $\mathcal{C}_{wx}^{alt}(k)$. (Compare with the spaces $\mathcal{E}_v(k)$).

3. For $\alpha \in (R')^{\perp\perp}$ the system $\mathcal{C}_x^{alt}(k)$ has exponents 0 and $1-2k_\alpha$ with multiplicities $m-1$ and 1 respectively along $\alpha^\perp \cap V^{R'}$.

4. Suppose $S \in \mathcal{R}_o(R',R)$ and $rk(S) = n-1$. Then the local exponents along $\mathcal{H}^S$ are $\nu(R',k)$ and $\nu(R',k)-\nu(S,k)$ with multiplicities $m-1$ and 1 respectively.

Proof: Clearly a germ in $\mathcal{C}_x^{alt}(k)$ is homogeneous of degree

$$\nu(R,k) + \sum_{S\in\mathcal{S}_o(R',R)} (\nu(R',k)-\nu(S,k)).$$

Property 1 follows by using theorem 4.6. Translation of a germ in $\mathcal{C}_x^{alt}(k)$ to $wx \in \mathcal{H}^{R'}$ yields a germ in $\mathcal{C}_{wx}^{alt}(k)$ by the properties of the system $\mathcal{E}_v(k)$. This proves 2. Let $j > n-m$ be such that $\alpha_j \in V^{R'}$. Then $\rho(k,g_j)$ commutes with all $\rho(k,g_i)$, $i \leq n-m$, and hence any special eigenvector of $\rho(k,g_j)$ is $\rho(k,g_i)$-invariant for $i \leq n-m$. This proves 3. Suppose S is as in 4. We may assume that S is generated by the simple roots $\alpha_1,\ldots,\widehat{\alpha_j},\ldots,\alpha_n$ for some $j > n-m$. The element $\rho(k,g_1\cdots\widehat{g_j}\cdots g_n)^{h(S)}$, $h(S)$ the Coxeter number of $W(S)$, commutes with all $\rho(k,g_i)$, $i \leq n-m$. Hence an eigenvector of this element with eigenvalue one (unique upto scalar multiples) is kept fixed by all $\rho(k,g_i)$, $i \leq n-m$. So the exponents along $\mathcal{H}^S$ are

$$\mu + \sum_{T\in\mathcal{S}_o(R',S)} (\nu(R',k)-\nu(T,k)),\ \mu \in \{\nu(S,k),0\}$$

with multiplicities $m-1$ ($\mu = \nu(S,k)$) and 1 ($\mu = 0$). Property 4 follows by applying theorem 4.6. $\square$

Now push forward the system $\mathcal{C}_x^{alt}(k)$ by γ to obtain the space $\mathcal{C}_{\gamma(x)}^{cone}(k)$ of germs at $y := \gamma(x) \in \gamma(\mathcal{H}^{R'})$. For this system one has the following (compare with the previous lemma).

Lemma 4.11 *1. Any germ in $\mathcal{C}_y^{cone}(k)$ is homogeneous of degree $-\nu(R',k)$.*

2. *For $w \in W(R',R)$ there is a canonical isomorphism of the vector space $\mathcal{C}_y^{cone}(k)$ onto $\mathcal{C}_{w.y}^{cone}(k)$.*

3. *Let $\alpha \in (R')^{\perp\!\!\perp}$. The exponents of $\mathcal{C}_y^{cone}(k)$ along the fixed points Γ_α of the reflection s_α are 0 and $1-2k_\alpha$ with multiplicities $m-1$ and 1 respectively.*

4. *Let $S \in \mathcal{R}_o(R',R)$ such that $rk(S) = n-1$. The exponents of $\mathcal{C}_y^{cone}(k)$ along $\mathcal{G}_S$ are 0 and $-\nu(S,k)$ with multiplicities $m-1$ and 1 respectively.*

Proof: The map γ is homogeneous of degree -1, hence the degree of $\mathcal{C}_y^{cone}(k)$ equals minus the degree of $\mathcal{C}_x^{alt}(k)$. This proves 1. Properties 2 and 3 are clear. Suppose S is as in 4. Take $v \in \mathcal{H}^S$ and $\xi \in V^{R'}$ such that $v+\epsilon\xi \in \mathcal{H}^{R'}$ for small $\epsilon \neq 0$. The exponents in property 4 can be derived by considering the smooth curve $\gamma(\varepsilon^{-1}v+\xi)$, ε small, passing through $\mathcal{G}_S$ together with the exponents and homogeneous degree of $\mathcal{C}_x^{alt}(k)$. $\Box$

Let A be the algebra of $W(R',R)$-invariants in the coordinate ring of the affine variety $\mathrm{Cone}(R',R)$. Let A^+ be the maximal ideal of elements with vanishing constant term. Take $\mathrm{Cone}(R',R)/W(R',R) := \mathrm{Spec}(A)$ and think of this as a weighted homogeneous affine variety. Then $A^+ \in \mathrm{Spec}(A)$ corresponds to 0 in this variety and we call this the origin of $\mathrm{Cone}(R',R)/W(R',R)$.

For a homogeneous set U we write $\Gamma^W(U)$ for the image of $\Gamma(U)/W(R',R)$ in $\mathrm{Spec}(A)$. The space $\mathrm{Spec}(A)$ has a natural stratification induced by the intersection structure of the codimension one subspaces $\Gamma_S^W := \Gamma_S/W(R',R)$ and $\Gamma_\alpha^W := \Gamma_\alpha/W(R',R)$. Here S ranges over the elements in $\mathcal{R}_o(R',R)$ of rank $n-1$ and α ranges over $(R')^{\perp\!\!\perp}$.

Let $Y \subset \gamma(\mathcal{H}^{R'})$ denote the $W(R',R)$-orbit of y. As in the case of the system $\mathcal{E}_v(k)$, the system $\mathcal{C}_y^{cone}(k)$ gives rise to a m-dimensional system on $\Gamma^W(\mathcal{H}^{R'})$ (a *smooth* subvariety) by property 2. Denote this system by $\mathcal{C}_Y^{cone}(k)$. Again, monodromy induces a representation ρ^* of the fundamental group $\pi_1(\Gamma^W(\mathcal{H}^{R'}),Y)$ on the dual $\mathcal{C}_Y^*(k)$ of $\mathcal{C}_Y^{cone}(k)$.

Lemma 4.12 *Assume that the parameter $k \in K'_{>0}$ is chosen in such a way that both $\nu(R,k)$ and $\nu(R',k)$ are in the hyperbolic range. Then there exists a positive definite ρ^*-invariant Hermitian form on $\mathcal{C}_Y^*(k)$.*

Proof: The m-dimensional subspace of $\mathcal{E}_{W(R)v}(k)$ fixed by the first $n-m$ reflections $\rho(k,g_j)$ is the orthoplement of the span of special eigenvectors $e_1(k)$

upto $e_{n-m}(k)$ with respect to the monodromy invariant hyperbolic form. By assumption the form restricted to this span is also hyperbolic and hence it is definite on the orthoplement. It is also invariant on the altered system and on its push forward. This proves the lemma. □

We are now in a position to prove the main theorem of this section. Suppose $S \in \mathcal{R}_o(R', R)$ has rank $n-1$ and $\alpha \in (R')^{\perp\!\!\perp}$. Define $p_\alpha := 2/(1-2k_\alpha)$ and $p_S := -z/\nu(S,k)$ where z is either 1 or 2 depending on whether or not $W(R', R)$ contains an element that acts as an involution fixing $\mathcal{G}_S$.

Theorem 4.7 *Assume that both $\nu(R,k)$ and $\nu(R',k)$ are in the hyperbolic range (as in lemma 4.12). Assume $p_\alpha \in \mathbb{N}_{\geq 2}$ and $p_S \in \mathbb{N}_{\geq 1}$ for all p_α and p_S defined above.*

Let $X_u(p) \to \Gamma^W(\mathcal{H}^{R'})$ be the universal Galois covering of local degrees p_α and p_S along Γ^W_α and Γ^W_S respectively. Then $X_u(p)$ embeds in a ramified covering $X_r(p)$ of Spec(A). Moreover $X_r(p)$ naturally carries the structure of a vector space and the covering automorphism group is a finite group of linear transformations.

Proof: The proof is based on essentially the same ideas found in the proof of theorem 3.14, page 58. Again $\mathcal{C}^{cone}_Y(k)$ induces a canonical multivalued *evaluation* map ev from $\Gamma^W(\mathcal{H}^{R'})$ into the dual $\mathcal{C}^*_Y(k)$.

The covering $X_u(p)$ extends to a ramified covering $X^o_r(p)$ over the relative interiors of the codimension one divisors. By a computation of the Wronskian of $\mathcal{E}_v(k)$ similar to the one in the proof of theorem 3.13, page 56 one can prove that the evaluation map lifts to a single valued *immersion* ev^o_r on $X^o_r(k)$.

Now one proceeds by induction on the corank of R' in R. Let x be a point on $\mathcal{G}_S$ for some $S \in \mathcal{R}_o(R', R)$ of rank $\text{rk}(R') + m$ for some $m > 0$. By theorem 4.2 x has a neighborhood U which is isomorphic to the product

$$\Delta^m \times (\Delta^{N(S,R)} \cap \text{Cone}(S,R)).$$

Now by assumption the stabilizer of x in $W(R', R)$ is a direct product of $W(S, R)$ and the subgroup of $W(S)$ generated by all reflections fixing R' and x.

In particular the factors in this direct product each act in a seperate factor in the Cartesian product for U written above. Hence the projection of x on Spec(A) has a small neighborhood whose intersection with the regular part $\Gamma^W(\mathcal{H}^{R'})$ is also a product $U_1 \times U_2$. Here U_1 is the complement of the discriminant of a finite reflection group in a neighborhood of 0 and U_2 is the regular part of Cone$(S,R)/W(S,R)$ intersected with a neighborhood of its origin.

By the induction hypothesis and the results of section 3.5 one concludes that $X^o_r(p)$ embeds in a ramified covering $X^*_r(p)$ of Spec$(A)\backslash\{A^+\}$ and $X^*_r(p)$ is a *smooth* variety.

The map ev_r^o extends locally biholomorphically over $X_r^*(p)$ to a map ev_r^*. The fact that monodromy of ev admits a positive definite invariant Hermitian form implies that this extension is an *isomorphism* onto $\mathcal{C}_Y^*(k)\backslash\{0\}$.

Then $\mathcal{C}_Y^*(k)$ is a ramified covering of Spec(A) extending $X_u(p)$ and its automorphism group is just the monodromy group of ev. □

Remark 4.4 *The condition that all stabilizers should be direct products is not strictly necessary. The proof of the main theorem 4.15 in the next section is more general. The argument given there could be applied here as well. It turns out however that we do not need the stronger result that would be obtained.*

Remark 4.5 *With the given assumptions theorem 4.7 implies that the homogeneous degree $-\nu(R', k)$ of $\mathcal{C}_Y^{cone}(k)$ equals z/m for some integer $m \geq 1$. Here z is either 1 or 2 depending on whether or not $W(R', R)$ contains an element acting as -1 on $Cone(R', R)$.*

4.4 GIT and root systems

In this section we generalize the usage of Geometric Invariant Theory as in [DM] to arbitrary root systems. The relation between our definitions and $\mathrm{SL}(2,\mathbb{C})$-invariants is explained in theorem 4.8.

Denote the polynomial algebra of V by $P[V]$ and let

$$P[V] = \bigoplus_{d\geq 0} P^d[V]$$

be its canonical grading in homogeneous components. If $V = V_1 \oplus V_2$ for two linear subspaces V_1, V_2 then there is a canonical isomorphism

$$P^d[V] \cong \bigoplus_{p+q=d} P^p[V_1] \otimes P^q[V_2].$$

We will consider an element of the space $P^p[V_1] \otimes P^q[V_2]$ as a $P^p[V_1]$-valued polynomial on V_2 homogeneous of degree q. In particular for any $S \in \mathcal{R}_o(\emptyset, R)$ we have such a decomposition arising from $V = V_S \oplus V^S$.

Definition 4.3 *Let $S' \subseteq S$ be two elements of $\mathcal{R}_o(\emptyset, R)$. Let*

$$P \in P^a[V_{S'}] \otimes P^b[V^{S'}],$$

i.e. a $P^a[V_{S'}]$-valued polynomial on $V^{S'}$. If $P \neq 0$ the vanishing multiplicity *of P along V^S is defined by*

$$\mathfrak{m}_S(P) := max\{j \in \mathbb{N} \mid P \in \bigoplus_{d\geq j} P^a[V_{S'}] \otimes P^d[V_S \cap V^{S'}] \otimes P^{b-d}[V^S]\}.$$

It is useful to define $\mathfrak{m}_S(0) = \infty$ with $\infty > m$ for all integers m. The projection of P in

$$P^{a+\mathfrak{m}_S(P)}[V_S] \otimes P^{b-\mathfrak{m}_S(P)}[V^S]$$

is a $P^{a+\mathfrak{m}_S(P)}[V_S]$-valued polynomial on V^S and will be denoted by P_S.

Note that if S' has corank one in S and $\lambda \in V_S \cap V^{S'}$, $\lambda \neq 0$ and

$$P \in P^a[V_{S'}] \otimes P^b[V^{S'}]$$

then P is divisible by $(\lambda^*)^{\mathfrak{m}_S(P)}$ and no higher power of λ^*. Here λ^* denotes the linear functional $(\cdot, \lambda)$ on $V^{S'}$.

Lemma 4.13 *Let $S'' \subseteq S' \subseteq S$ be three elements of $\mathcal{R}_o(\emptyset, R)$ and*

$$P \in P^a[V_{S''}] \otimes P^b[V^{S''}].$$

Then the following inequality between multiplicities holds:

$$\mathfrak{m}_S(P_{S'}) \geq \mathfrak{m}_S(P) - \mathfrak{m}_{S'}(P)$$

Proof: This follows from the decomposition

$$V^{S''} \cap V_S = (V_{S'} \cap V^{S''}) \oplus (V^{S'} \cap V_S).$$

Indeed if $P_{S'}$ has a non zero component in

$$P^a[V_{S''}] \otimes P^{d_1}[V_{S'} \cap V^{S''}] \otimes P^{d_2}[V^{S'} \cap V_S] \otimes P^{d_3}[V^S]$$

then $d_1 = \mathfrak{m}_{S'}(P)$ and $d_1 + d_2 \geq \mathfrak{m}_S(P)$. Hence $d_2 \geq \mathfrak{m}_S(P) - \mathfrak{m}_{S'}(P)$ which implies the same lower bound for $\mathfrak{m}_S(P_{S'})$. $\square$

We can now introduce the key object for the construction, a certain algebra of polynomials. It is convenient to define $\nu(\emptyset, k) := 1$.

Let k be a rational multiplicity parameter such that $\nu(R, k)$ lies in the range $(1-m_2, 0]$, i.e. is of hyperbolic or parabolic type (m_2 denotes the second smallest exponent of R). Let $N > 0$ be a common denominator of the k_α, i.e. $Nk_\alpha \in \mathbb{Z}$ for all $\alpha \in R$. Then for any root subsystem $S \in \mathcal{R}_o(\emptyset, R)$ we have $N\nu(S, k) \in \mathbb{Z}$. Indeed any $\nu(S, k)$ is an affine function in k with integer coefficients.

Definition 4.4 *We define a vector space*

$$A_N(R, k) := \{P \in P^{-N\nu(R,k)}[V] \mid \mathfrak{m}_S(P) \geq -N\nu(S, k) \text{ for all } S \in \mathcal{R}_o(\emptyset, R) \text{ of rank } rk(R) - 1\}$$

Define a $\mathbb{C}$-algebra $\mathcal{A}_N(R, k)$ by

$$\mathcal{A}_N(R, k) := \sum_{d \geq 0} A_{dN}(R, k)$$

If $\nu(R, k) < 0$ then this algebra has a natural grading (the sum in its definition is then a direct sum). If $\nu(R, k) = 0$ then $\mathcal{A}_N(R, k) \cong \mathbb{C}$.

Before studying the structure of this algebra we first show its relation to Geometric Invariant Theory as encountered in chapter 2. Take R of type A_n. Use the standard realisation of this root system in $\mathbb{C}^{n+1}$. Let $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{C}^{n+1}$ and V the n-dimensional subspace of all vectors for which the sum of their coordinates equals zero. The roots are then given by $e_i - e_j$ for $1 \leq i, j \leq n+1$ and $i \neq j$.

Let $k = m/N$ for positive integers m and N such that $N < (n+1)m < 2N$. Define m_{n+2} as the remainder $2N - (n+1)m$ and take $m_j := m$ for $j \in \{1, \ldots, n+1\}$. Let π_j denote the canonical projection of $(\mathbb{P}^1)^{n+2}$ onto the j^{th} factor $\mathbb{P}^1$ for $j = 1, \ldots, n+2$. Define a line bundle $\mathcal{L}$ over $(\mathbb{P}^1)^{n+2}$ as follows

$$\mathcal{L} := \bigotimes_{j=1}^{n+2} \pi_j^* \mathcal{O}_{\mathbb{P}^1}(m_j).$$

Consider the diagonal $\mathrm{SL}(2,\mathbb{C})$-action on $(\mathbb{P}^1)^{n+2}$. Then there is a canonical $\mathrm{SL}(2,\mathbb{C})$-action on $\mathcal{L}$ turning it into a homogeneous line bundle. We can identify global sections in $\mathcal{L}^{\otimes d}$ with polynomials in $2n+4$ variables (written in matrix form)

$$P \begin{pmatrix} x_{1,1} & \ldots & x_{1,n+2} \\ x_{2,1} & \ldots & x_{2,n+2} \end{pmatrix}$$

that are homogeneous of degree dm_j in the j^{th} column. An element $g \in \mathrm{SL}(2,\mathbb{C})$ acts on such a section by

$$(gP)(\mathbf{x}) := P(g^{-1}\mathbf{x})$$

for a matrix $\mathbf{x}$ and matrix multiplication in the right hand side argument.

Theorem 4.8 *There is an isomorphism of algebras*

$$A(\mathcal{L}) := \bigoplus_{d \geq 0} \Gamma((\mathbb{P}^1)^{n+2}, \mathcal{L}^{\otimes d})^{\mathrm{SL}(2,\mathbb{C})} \cong \mathcal{A}_N(A_n, k).$$

Here the left hand side is the graded algebra of invariant sections in powers of $\mathcal{L}$.

Proof: Define a linear map γ_d of $\Gamma((\mathbb{P}^1)^{n+2}, \mathcal{L}^{\otimes d})^{\mathrm{SL}(2,\mathbb{C})}$ into $P^{-dN\nu(A_n,k)}[V]$ by

$$\gamma_d(P)(x_1, \ldots, x_{n+1}) := P \begin{pmatrix} x_1 & \ldots & x_{n+1} & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}$$

Indeed one can check that $\gamma_d(P)$ is homogeneous of degree $-dN\nu(A_n, k)$ by considering the action of $\mathrm{diag}(\lambda, \lambda^{-1})$ for $\lambda \in \mathbb{C}^*$.

The $\mathrm{SL}(2,\mathbb{C})$-orbit $\mathcal{O}$ of the set

$$\{ \begin{pmatrix} x_1 & \ldots & x_{n+1} & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix} \mid (x_1, \ldots, x_{n+1}) \in V \}$$

is just

$$\{\mathbf{y} \in \mathrm{Mat}(2 \times (n+2), \mathbb{C}) \mid \det(y_j \; y_{n+2}) \neq 0 \text{ for all } j \leq n+1\}$$

where y_j denotes the j^{th} column of $\mathbf{y}$. In particular this set is *dense* and hence γ_d is *injective*. Remains to compute its image.

A parabolic irreducible root subsystem of A_n is of type A_s for some $s \leq n$. If S is such a system of rank $s < n$ then there exists a subset

$$I \subset \{1, \ldots, n+1\}, \; \#I = s+1$$

such that V^S is just the set of all vectors in V whose coordinates e_j, $j \in I$ coincide. The reflection group $W(A_n)$ acts transitively on parabolic root subsystem hence we may assume that I is $\{1, \ldots, s+1\}$. Let $(x, \ldots, x, x_{s+2}, \ldots, x_{n+1}) \in V^S$ and $x_1, \ldots, x_{s+1} \in \mathbb{C}$ such that $\sum_{j=1}^{s+1} x_j = 0$. If P is an invariant section in $\Gamma((\mathbb{P}^1)^{n+2}, \mathcal{L}^{\otimes d})$ then an elementary calculation yields for all $\lambda \in \mathbb{C}$:

$$P \begin{pmatrix} x+\lambda x_1 & \ldots & x+\lambda x_{s+1} & x_{s+2} & \ldots & x_{n+1} & 1 \\ 1 & \ldots & & & \ldots & 1 & 0 \end{pmatrix} =$$

$$\lambda^{-dN\nu(A_s,k)} P \begin{pmatrix} x_1 & \ldots & x_{s+1} & x_{s+2}-x & \ldots & x_{n+1}-x & 1 \\ 1 & \ldots & 1 & \lambda & \ldots & \lambda & 0 \end{pmatrix}$$

This shows that $\mathfrak{m}_S(\gamma_d(P)) \geq -dN\nu(A_s, k)$ and in particular $\gamma_d(P)$ is an element of $A_{dN}(A_n, k)$. On the other hand if $P \in A_{dN}(A_n, k)$ one can define a function $\widetilde{P}$ on the dense orbit $\mathcal{O}$ by prescribing

$$\widetilde{P} \begin{pmatrix} x_1 & \ldots & x_{n+1} & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix} := P(x_1, \ldots, x_{n+1})$$

and extending it by $\mathrm{SL}(2, \mathbb{C})$-invariance and homogeneity properties. The conditions on the vanishing multiplicities of P are easily seen to imply that $\widetilde{P}$ is locally bounded near any point in $\mathrm{Mat}(2 \times (n+2), \mathbb{C})$. Hence $\widetilde{P}$ extends to an invariant section also denoted $\widetilde{P}$. Clearly $\gamma_d(\widetilde{P}) = P$. The sequence $(1, \gamma_1, \gamma_2, \ldots)$ gives the isomorphism. $\Box$

Using theorem 4.6 on page 91 we derive two important facts about the algebra $\mathcal{A}_N(R, k)$.

Theorem 4.9 *If $P \in A_{dN}(R, k)$ and $S \in \mathcal{R}_o(\emptyset, R)$ then the following inequality holds:*

$$\mathfrak{m}_S(P) \geq -dN\nu(S, k)$$

Proof: This is true if $rk(S) \geq n-1$. Now use downward induction on the rank of S. So suppose the above inequality holds for any rank greater than m. Let $S \in$

$\mathcal{R}_o(\emptyset, R)$ be of rank m. For any $S' \in \mathcal{S}_o(S, R)$ the space $V^{S'}$ is of codimension one in V^S. P_S is a homogeneous polynomial of degree $-dN\nu(R,k) - \mathfrak{m}_S(P)$ on V^S. Moreover, by our induction hypothesis and lemma 4.13 we have:

$$\mathfrak{m}_{S'}(P_S) \geq -dN\nu(S', k) - \mathfrak{m}_S(P)$$

This leads to the equation:

$$\sum_{S' \in \mathcal{S}_o(S,R)} (-dN\nu(S', k) - \mathfrak{m}_S(P)) \leq -dN\nu(R, k) - \mathfrak{m}_S(P)$$

Now using the equality from theorem 4.6 this is equivalent to

$$\mathfrak{m}_S(P) \geq -dN\nu(S, k).$$

The theorem follows by induction. □

When equality holds, one can make a sharper statement.

Theorem 4.10 *Let $S \subset R$ be a parabolic irreducible root subsystem. If $P \in A_{dN}(R, k)$ and $\mathfrak{m}_S(P) = -dN\nu(S, k)$ then P_S is a pure product:*

$$P_S = Q \otimes \prod_{S' \in \mathcal{S}_o(S,R)} (\beta^*_{S'})^{-dN(\nu(S',k) - \nu(S,k))}$$

Here $\beta_{S'} \in V_{S'} \cap V^S$ is a non-zero vector and Q is an element of $A_{dN}(S, k)$.

Proof: If $\mathfrak{m}_S(P) = -dN\nu(S, k)$ then it follows from theorem 4.9 and lemma 4.13 that for any $S' \in \mathcal{S}_o(S, R)$ the following inequality holds:

$$\mathfrak{m}_{S'}(P_S) \geq -dN(\nu(S', k) - \nu(S, k))$$

Hence P_S is divisible by $(\beta^*_{S'})^{-dN(\nu(S',k)-\nu(S,k))}$. This implies that P_S is divisible by a product of linear factors of total degree at least $-dN(\nu(R, k) - \nu(S, k))$. But this is exactly the homogeneous degree of P_S. This shows that it equals a Q times this product of linear functions for some $Q \in P^{-dN\nu(S,k)}[V_S]$. That this coefficient Q is in fact in $A_{dN}(S, k)$ follows from the decomposition

$$V = V_{S'} \oplus (V^{S'} \cap V_S) \oplus V^S$$

for any $S' \in \mathcal{R}_o(\emptyset, S)$. □

Note that the proof of theorem 4.10 even shows that the equality $\mathfrak{m}_S(P) = -dN\nu(S, k)$ implies the equality $\mathfrak{m}_{S'}(P) = -dN\nu(S', k)$ for all $S' \in \mathcal{S}_o(S, R)$ and hence for all $S' \in \mathcal{R}_o(S, R)$.

The next lemma shows an interrelation between exponents of irreducible parabolic root subsystems.

Lemma 4.14 *Suppose $\nu(R,k)$ is of hyperbolic type, i.e. $1-m_2 < \nu(R,k) < 0$. Let $S \in \mathcal{R}_o(\emptyset, R)$ such that $\nu(S,k)$ is also of hyprbolic type. Let $S' \in \mathcal{R}(S,R)$. If S' is irreducible then $\nu(S',k)$ is of hyperbolic type. If S' is reducible and S'' is an irreducible component of S' not containing S then $\nu(S'',k) > 0$ i.e. it is of elliptic type.*

Proof: Because both are parabolic we may assume that S and S' are generated by simple roots of R. Say by simple roots α_j for $j \in I$ or $j \in I'$ respectively.

Assume that S' is irreducible. Then the hyperbolic Hermitian form $H(k)$ as defined in section 3.2 restricts to a hyperbolic form on the $\mathbb{C}$-span of e_j, $j \in I$. Hence its restriction to the bigger $\mathbb{C}$-span of e_j, $j \in I'$ must also be hyperbolic. This implies that $\nu(S',k)$ is of hyperbolic type.

If S' is reducible and S'' is an irreducible component of S' different from S then S'' is generated by simple roots α_j, $j \in I''$. Moreover $\alpha_i \perp \alpha_j$ if $i \in I$ and $j \in I''$. By definition of $H(k)$ the space $\mathrm{Span}_{\mathbb{C}}\{e_i \mid i \in I\}$ is perpendicular to $\mathrm{Span}_{\mathbb{C}}\{e_j \mid j \in I''\}$ with respect to $H(k)$. Because $H(k)$ is hyperbolic on the former span it must by positive definite (elliptic) on the latter. This implies that $\nu(S'',k) > 0$. $\square$

Theorem 4.10 allows the following important construction of algebra homomorphisms from $\mathcal{A}_N(S,k)$ to $\mathcal{A}_N(S',k)$ for $S' \subseteq S$.

Definition 4.5 *Let $S' \subseteq S \subseteq R$ be irreducible and parabolic root subsystems such that $\nu(S',k) \leq 0$. Define a homomorphism $\phi_{S',S} : \mathcal{A}_N(S,k) \to \mathcal{A}_N(S',k)$ of graded algebras as follows. Fix a polynomial on $V^{S'} \cap V_S$ given by*

$$\Pi := \prod_{S'' \in \mathcal{S}_o(S',S)} (\beta^*_{S''})^{-N(\nu(S'',k)-\nu(S',k))}$$

as in theorem 4.10 for $d=1$. If $P \in A_{dN}(S,k)$ then $\phi_{S',S}(P)$ is defined as the projection of P on

$$P^{-dN\nu(S',k)}[V_{S'}] \otimes P^{-dN(\nu(S,k)-\nu(S',k))}[V^{S'} \cap V_S]$$

divided by Π^d.

Call two homomorphisms ϕ_1, ϕ_2 between graded $\mathbb{C}$-algebras A_1, A_2 *equivalent* if there exists a non zero complex number t such that for any homogeneous $z \in A_1$

$$\phi_1(z) = t^{\deg(z)} \phi_2(z).$$

Note that the construction of $\phi_{S',S}$ is unique upto equivalence of homomorphisms. These homomorphisms relate nicely to each other according to the following lemma.

Lemma 4.15 *For any sequence $S'' \subseteq S' \subseteq S$ the homomorphisms*

$$\phi_{S'',S'} \circ \phi_{S',S} \text{ and } \phi_{S'',S}$$

are equivalent.

Proof: Take $P \in A_{dN}(S,k)$. If $\mathfrak{m}_{S''}(P) > -dN\nu(S'',k)$ then both homomorphisms are zero at P. So suppose $\mathfrak{m}_{S''}(P) = -dN\nu(S'',k)$ and hence $\mathfrak{m}_{S'}(P) = -dN\nu(S',k)$. Using these two equalities and applying theorem 4.10 twice shows that P has a non zero component

$$Q \otimes \Pi' \otimes \Pi'' \in P^a[V_{S''}] \otimes P^b[V^{S''} \cap V_{S'}] \otimes P^c[V^{S'}]$$

$$a = -dN\nu(S'',k),\ b = -dN(\nu(S',k) - \nu(S'',k)),$$
$$c = -dN(\nu(S,k) - \nu(S',k)).$$

Here Π' and Π'' are products of linear factors and $Q \in A_{dN}(S'',k)$ divides P. In particular both homomorphisms map P to a non zero scalar multiple of Q. $\square$

Remark 4.6 *One can even prove that the $\phi_{S',S}$ can be constructed in such a way that the homomorphisms $\phi_{S'',S'} \circ \phi_{S',S}$ and $\phi_{S'',S}$ are equivalent by a "twist" of* ± 1.

Now we construct a variety that is the affine cone of a completion of the projective set $\mathbb{P}(V^{reg})$ depending on the multiplicity parameter k. The variety Q^{sst} appearing in [DM] is the $\mathrm{SL}(2,\mathbb{C})$-quotient of $(\mathbb{P}^1)^{n+2}$ with respect to the line bundle $\mathcal{L}$, i.e. $\mathrm{Proj}(A(\mathcal{L}))$. To get a completion of $\mathbb{P}(V^{reg})$ in general it is reasonable to consider

$$\mathrm{Proj}(\mathcal{A}_N(R,k)).$$

However there remain some problems that complicate the study of this space in great detail. Here are some important ones.

1. For what parameters k is $\mathcal{A}_N(R,k)$ non-trivial?
2. Is $\mathcal{A}_N(R,k)$ finitely generated?
3. Are the homomorphisms $\phi_{S',S}$ as introduced before *surjective*?
4. Does $\mathbb{P}(V^{reg})$ embed as an open dense set?

Of course if R is of type A_n then these questions can be answered affirmatively. The case of general systems remains unclear. At the end of this chapter I present some partial results on the stated questions. To do so we consider only a subalgebra of $\mathcal{A}_N(R,k)$ in that section. Namely the algebra generated by products of dual roots.

Now instead of studying the algebra $\mathcal{A}_N(R,k)$ I consider the subalgebra generated by one homogeneous component $A_N(R,k)$. This has the advantage that the corresponding affine variety can be constructed in a straight forward way resulting in explicit formulas and computations.

Now fix an irreducible root system R of full rank ≥ 3 in E and a rational multiplicity parameter k such that $\nu(R,k)$ is of hyperbolic type. For any $S \in \mathcal{R}_o(\emptyset, R)$ we define

$$\mathcal{H}_o^S = \mathcal{H}_o^S(R) := V^S \backslash \cup \{V^{S'} \mid S' \in \mathcal{R}_o(S,R) \text{ with } \nu(S',k) \leq 0\}$$

Note that this coincides with $\mathcal{H}_o^S(R)$ of section 4.3 if $\nu(S,k) \leq 0$ so notation should not be too confusing.

Fix a common denominator $N > 0$ of k such that $N\nu(S,k)$ is *even* for all $S \in \mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) < 0$. The space $A_N(R,k)$ is clearly a finite dimensional vector space. If $A'_N(R,k)$ is its dual we denote the canonical map of V_R into $A'_N(R,k)$ (evaluation) by ι_R. Then ι_R is homogeneous of degree $-N\nu(R,k)$ in particular $\iota_R(-v) = \iota_R(v)$ for all $v \in V_R$.

Like Γ^o in section 4.3, page 87 we define

$$I_R^o := \{(v,y) \in V_R \times A'_N(R,k) \mid \iota_R(v) = \lambda y \text{ for some } \lambda \in \mathbb{C}^*\}.$$

Let I_R be the closure of I_R^o and define $I_R(Y)$ for a subset $Y \subseteq V_R$ by

$$I_R(Y) := \{y \in A'_N(R,k) \mid (v,y) \in I_R \text{ for some } v \in Y\}.$$

The homogeneous affine variety $Q(R,k)$ is by definition $I_R(\{0\})$, i.e. the closure of $\iota_R(V_R)$.

Note that if $\nu(R,k) = 0$ then $Q(R,k) \cong \mathbb{C}$ and ι_R is a constant non zero map.

To relate the varieties $Q(R,k)$ and $\mathrm{Cone}(R',R)$ we need the following theorem.

Theorem 4.11 *Let $S \in \mathcal{R}_o(\emptyset, R)$. Define a sequence of $N(S,R)$ polynomials on V^S as follows:*

$$F_{S'} := \prod_{S'' \neq S'} \beta^*_{S''}, \ S' \in \mathcal{S}_o(S,R)$$

The product is taken over every $S'' \in \mathcal{S}_o(S,R)$ and $\beta_{S''}$ is a fixed non zero vector in $V_{S''} \cap V^S$ for all $S'' \in \mathcal{S}_o(S,R)$.

If P is a polynomial on V^S such that:

1. *P is homogeneous of degree $m(N(S,R)-1)$ for some $m \geq 1$.*
2. *For all $S' \in \mathcal{R}_o(S,R)$ the vanishing multiplicity of P along $V^{S'}$ satisfies*

$$\mathfrak{m}_{S'}(P) \geq m(N(S,S')-1).$$

Then there exists a polynomial Q in the indeterminates $X_{S'}$, $S' \in \mathcal{S}_o(S,R)$ such that Q is homogeneous of degree m and

$$P = Q((F_{S'})_{S' \in \mathcal{S}_o(S,R)}).$$

Proof: The proof is given by using a partial fraction decomposition theorem. It will appear in a separate article. $\square$

This theorem has the following important consequence.

Theorem 4.12 *Suppose $S \in \mathcal{R}_o(\emptyset, R)$ and $\nu(S,k) < 0$. Let γ $(= \gamma_{S,R})$ denote the map of $\mathcal{H}_o^S$ into Cone(S,R) as in section 4.3. Let for each $S' \in \mathcal{S}_o(S,R)$ the corresponding coordinate of γ be given by $1/\beta^*_{S'}$. The map*

$$\Phi_S : V_S \times \gamma(\mathcal{H}_o^S) \to Q(R,k)$$

given by

$$\Phi_S : (v,y) \mapsto \left(\prod_{S' \in \mathcal{S}_o(S,R)} y_{S'}^{-N(\nu(S',k)-\nu(S,k))} \right) \cdot \iota_R(v + \gamma^{-1}(y))$$

is the restriction of a polynomial map on $V_S \times \mathbb{C}^{N(S,R)}$. In particular it extends to a morphism Φ_S of $V_S \times$ Cone(S,R) into $Q(R,k)$. Moreover there exists a non zero constant c^{st} such that $\Phi_S(v,0) = c^{st} \iota_S(v) \circ \phi_{S,R}$ for all $v \in V_S$.

Proof: Let $P \in A_N(R,k)$ and let $d := -N\nu(S,k) + m$ for some $m \geq 0$. Denote the projection of P onto $P^d[V_S] \otimes P^{-N\nu(R,k)-d}[V^S]$ by P_d. Then P_d is divisible by the product

$$\Pi_d := \prod_{S' \in \mathcal{S}_o(S,R)} (\beta^*_{S'})^{-N\nu(S',k)-d}.$$

Consider the $P^d[V_S]$-valued polynomial P_d/Π_d on V^S. It is homogeneous of degree $m(N(S,R)-1)$ and satisfies $\mathfrak{m}_{S'}(P_d/\Pi_d) \geq m(N(S,S')-1)$ for any $S' \in \mathcal{R}_o(S,R)$. Now by putting the definitions together one checks

$$1/\Pi_0(\gamma^{-1}(y)) = \prod_{S' \in \mathcal{S}_o(S,R)} y_{S'}^{-N(\nu(S',k)-\nu(S,k))}.$$

Using theorem 4.11 we conclude that

$$\prod_{S' \in \mathcal{S}_o(S,R)} y_{S'}^{-N(\nu(S',k)-\nu(S,k))} P_d(u + \gamma^{-1}(y))$$

is polynomial in u and y and homogeneous of degree m in y. This shows that Φ_S extends to a morphism on $V_S \times \mathrm{Cone}(S,R)$. For $y = 0$ we get $\Phi_S(v,0)(P) =$

$(P_0/\Pi_0)(v)$ and this is exactly $\iota_S(v)\circ\phi_{S,R}(P)$ for all P upto some scalar multiple. $\square$

If $S \in \mathcal{R}_o(\emptyset, R)$ and $\nu(S,k) = 0$ then Φ_S has the invariance property

$$\Phi_S(\lambda u, \lambda^{-1} y) = \Phi_S(u, y)$$

for all $\lambda \in \mathbb{C}^*$. Moreover $\Phi_S(u,y)$ is a fixed point if either one of u and y vanishes (i.e. it does not depend on the other parameter). Let $\jmath_S$ be the canonical map of $V_S \times \mathbb{C}^{N(S,R)}$ into $V_S \otimes \mathbb{C}^{N(S,R)}$ (a Segre embedding from a projective geometric point of view). Then the map $\Phi_S^c := \Phi_S \circ \jmath_S^{-1}$ is a well defined morphism of $\jmath_S(V_S \times \mathrm{Cone}(S,R))$ into $Q(R,k)$.

At this point we will make some assumptions to assure that the varieties $Q(R,k)$ have some nice properties. Partial justification of these assumptions is given at the end of this section.

We make the following assumptions.

1. *For all $S \in \mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) < 0$: The map ι_S is an immersion of $\mathcal{H}_o^{\emptyset}(S)$ onto an open dense set in $Q(S,k)$. Moreover $\iota_S(v_1) = \iota_S(v_2)$ for some $v_1, v_2 \in V_S$ if and only if $v_1 = \omega v_2$ for some $\omega \in \mathbb{C}$ satisfying $\omega^{N\nu(S,k)} = 1$.*

2. *If $S \in \mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) \le 0$ and $S' \in \mathcal{R}_o(S,R)$ then $\phi_{S,S'}$ is surjective.*

3. *For any $S \in \mathcal{R}_o(\emptyset, R)$ such that $\nu(S,k) < 0$: If $v \in \mathcal{H}_o^{\emptyset}(S)$ then Φ_S is locally biholomorpic at $(v,0) \in V_S \times Cone(S,R)$, i.e. is the restriction of a locally biholomorphic map.*

4. *For any $S \in \mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) = 0$: The map*

$$\mathbb{C}^* \times \jmath_S(V_S \times Cone(S,R)) \ni (z,t) \mapsto z\Phi_S^c(t) \in Q(R,k)$$

   is locally biholomorphic at $(1,0)$.

If $S \subseteq S' \subseteq R$ are in $\mathcal{R}_o(\emptyset, R)$ the homomorphism $\phi_{S,R}$ induces an injective linear map $\phi_{S,R}^*$ of $A_N'(S,k)$ into $A_N'(R,k)$ and $\phi_{S',R}^* \circ \phi_{S,S'}^*$ equals $\phi_{S,R}^*$upto some scalar multiple. There is a nice relation between $Q(S,k)$ and $Q(R,k)$ using the map $\phi_{S,R}^*$.

Theorem 4.13 *Let $S \in \mathcal{R}_o(\emptyset, R)$ such that $S \ne R$ and $\nu(S,k) \le 0$. The map $\phi_{S,R}^*$ maps $Q(S,k)$ into $I_R(\mathcal{H}_o^S) \subset Q(R,k)$.*

Proof: Let $u_1 \in V_S$ and $u_2 \in \mathcal{H}_o^S$. Take $\mu \in \mathbb{C}^*$. Then $(\mu u_1 + u_2, \Phi_S(u_1, \mu\gamma(u_2))$ is an element of I_R where Φ_S and γ are as in theorem 4.12. In particular

$(u_2, \Phi_S(u_1, 0)) \in I_R$ which shows that $\Phi_S(u_1, 0) = c^{st}\phi^*_{S,R}\circ\iota_S(u_1)$ is contained in $I_R(\{u_2\})$. Because the latter set is closed $\phi^*_{S,R}$ maps $Q(S,k)$ into $I_R(\{u_2\})$. $\square$

Note that if $\nu(S,k) = 0$ then $\phi^*_{S,R}$ maps $Q(S,k)$ onto a *line* (a one dimensional linear space). Indeed $Q(S,k) \cong \mathbb{C}$ in this case. We call such lines the *cuspidal lines* on $Q(R,k)$ and any point on such a line a cuspidal point.

The following theorem proves that in fact $I_R(\mathcal{H}^S_o)$ is exactly the $\phi^*_{S,R}$-image of $Q(S,k)$.

Theorem 4.14 *The $\phi^*_{S,R}$-images of $I_S(\mathcal{H}^\emptyset_o(S))$ constitute a stratification of $Q(R,k)\backslash\{0\}$ if S ranges over all elements of $\mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) \leq 0$.*

Proof: Let $\pi : Y \to V_R\backslash\{0\}$ be a smooth blow up such that:

1. The restriction of π to $\pi^{-1}(\mathcal{H}^\emptyset_o)$ is an injective immersion.
2. For $S \in \mathcal{R}_o(\emptyset, R)$ with $\nu(S,k) \leq 0$ the closure of the preimage $\pi^{-1}(\mathcal{H}^S_o)$ in Y is a divisor of codimension one.
3. These divisors have normal crossings.

Take $n = \mathrm{rk}(R)$. Let $x \in Y$ be a point and $x_1, \ldots, x_n$ polydisc coordinates on a neighborhood U of x such that the exceptional divisors on Y passing through U have local equations $x_j = 0$, $j = 1, \ldots, s$ for some $s \leq n$. Let $W(R_j)$ be the stabilizer of the π-image of the divisor $x_j = 0$ for some root subsystem R_j. Define numbers $\mathfrak{m}_j$ as the multiplicity $-N\nu(R_j, k)$ for $j \leq s$.

Then the map on $U\backslash\{u \in U \mid x_j(u) = 0 \text{ for some } j \leq s\}$ given by

$$x_1^{-\mathfrak{m}_1} \cdots x_s^{-\mathfrak{m}_s} \cdot \iota_R\circ\pi$$

extends holomorphically over all points $u \in U$ with $x_j(u) = 0$ for at most one $j \leq s$. Indeed by the argument from the proof of the previous theorem it maps the set

$$\{u \in U \mid x_j(u) = 0 \text{ and } x_i(u) \neq 0 \text{ for all } i \leq s,\ i \neq j\}$$

into $\phi^*_{R_j,R}(Q(R_j,k))$. Now by Hartog's theorem the map extends over all of U and the divisor $x_j = 0$ necessarily gets mapped into $\phi^*_{R_j,R}(Q(R_j,k))$. The theorem follows by induction on the rank of R. $\square$

With this stratification of $Q(R,k)$ in mind, assumptions 3 and 4 about the nature of $Q(R,k)$ above give its local structure near non-cuspidal and cuspidal points respectively.

The reflection group $W(R)$ acts naturally on $A_N(R,k)$. The map ι_R is $W(R)$-equivariant and hence I_R is invariant under the diagonal $W(R)$-action. In particular the $W(R)$-action restricts to an action on $Q(R,k)$. The map $\phi^*_{S,R}$ is

$W(S)$-equivariant if we consider $W(S)$ as a subgroup of $W(R)$ in the natural way.

If $S \in \mathcal{R}_o(\emptyset, R)$ has rank $\mathrm{rk}(R)-1$ and $\nu(S,k) < 0$ then $I_R(\mathcal{H}_o^S)$ has codimension one in $Q(R,k)$.

Lemma 4.16 *For S as above the element $w \in W(R)$ acts as an involution on $Q(R,k)$ that fixes $I_R(\mathcal{H}_o^S)$ pointwise if and only if $\pm w$ is a reflection fixing S.*

Proof: By invariance of I_R and theorem 4.13 such an element w must act as a scalar ω on V_S for some ω satisfying $\omega^{N\nu(S,k)} = 1$. Then w maps S onto S and so $\omega = \pm 1$. Now w acts non trivial on $Q(R,k)$ and hence ωw has to be a reflection of V (S has corank one in R). $\square$

Any reflection in $W(R)$ acts as a certain involution on $Q(R,k)$ fixing a subvariety of codimension one. On V any subgroup $W(S)$ for $S \subset R$ a strict parabolic root subsystem has a nonzero simultaneous fixed point. On $Q(R,k)$ the situation is different.

Lemma 4.17 *Let $S \in \mathcal{R}_o(\emptyset, R)$. The subgroup $W(S)$ of $W(R)$ has a simultaneous fixed point on $\iota_R(\mathcal{H}_o^\emptyset)$ if and only if $\nu(S,k) > 0$.*

Proof: The map ι_R is $W(R)$-equivariant and its fibres are $C_{-N\nu(R,k)}$-orbits (cyclic group of roots of unity acting by scalar multiplication). Hence fixed points of a reflection $s_\alpha \in W(S)$ on $\iota_R(\mathcal{H}_o^\emptyset)$ are exactly the ι_R-images of its eigenspaces in V_R (recall that $N\nu(S,k)$ was supposed to be even). The intersection of eigenspaces of all reflections in $W(S)$ is exactly V^S. We conclude the proof by the observation that V^S intersects $\mathcal{H}_o^\emptyset$ if and only if $\nu(S,k) > 0$. $\square$

Corollary 4.4 *Let S be as in the previous lemma and let $S' \in \mathcal{R}_o(\emptyset, R)$ such that $\nu(S',k) < 0$. The group $W(S)$ has a simultaneous fixed point on the relative interior*

$$\phi^*_{S',R} \circ \iota_{S'}(\mathcal{H}_o^\emptyset(S'))$$

of $Q(S',k)$ embedded in $Q(R,k)$ if and only if $\nu(S,k) > 0$ and S' is $W(S)$-stable.

Proof: The set $\phi^*_{S',R} \circ \iota_{S'}(\mathcal{H}_o^\emptyset(S'))$ intersects no $I_R(\mathcal{H}_o^{S''})$ for any $S'' \subset S'$. By $W(R)$-invariance of I_R it follows that $\mathcal{H}_o^{S'}$ and hence S' must be $W(S)$-stable. If $S \perp S'$ then $\nu(S,k) > 0$ by lemma 4.14 on page 100 and $W(S)$ even fixes $I_R(\mathcal{H}_o^{S'})$ pointwise. If $S \subseteq S'$ we can apply the previous lemma on $Q(S',k)$ by $W(S)$-equivariance of $\phi^*_{S',R}$. $\square$

The importance of these observations is that if $x \in Q(R,k) \backslash \{0\}$ is any non cuspidal point then x can be $W(S)$-stable for some $S \in \mathcal{R}_o(\emptyset, R)$ only if $\nu(S,k) > 0$, i.e. is of elliptic type. This plays an important role in proving the main theorem on discreteness of monodromy in this case.

4.5 Hypergeometric functions

After studying the variety $Q(R,k)$ the hypergeometric function returns into play. In this section we finally want to prove discreteness of the monodromy group of the system $\mathcal{E}_S(k)$ under some natural integrality conditions on its exponents.

The hypergeometric system for a root system of type D_n is actually the same as that of type B_n if we define $k_\alpha = 0$ for the $2n$ "short" roots. Because of this and the fact that D_n plays an exceptional role in some sense (see remark 4.2 on page 86 for example) we do not consider root systems of type D_n in this section altogether.

The map ι_R is assumed to be an immersion on $\mathcal{H}_o^{\emptyset}$. In particular it is an immersion on V^{reg}. Consider the hypergeometric system $\mathcal{E}_v(k)$ of germs at the point $v \in V^{reg}$. Let $y := \iota_R(v)$ and denote the pushforward of $\mathcal{E}_v(k)$ by ι_R as $\mathcal{E}_y^Q(k)$. Naturally any germ in $\mathcal{E}_y^Q(k)$ can be continued analytically throughout $\iota_R(V^{reg})$. The system has the following properties.

Lemma 4.18 *1. The determination order of $\mathcal{E}_y^Q(k)$ is rk(R) and any determination is homogeneous of degree $1/N$.*

2. For any $w \in W(R)$ there is a canonical isomorphism of $\mathcal{E}_y^Q(k)$ onto $\mathcal{E}_{wy}^Q(k)$ as vector spaces.

3. For any root $\alpha \in R$ the system $\mathcal{E}_y^Q(k)$ has exponents 0 and $1-2k_\alpha$ along $\iota_R(\alpha^\perp \cap V^{reg})$ with multiplicities $n-1$ and 1 respectively.

4. Let $S \in \mathcal{R}_o(\emptyset, R)$ be of rank rk$(R)-1$ such that $\nu(S,k) < 0$. Then the exponents of $\mathcal{E}_y^Q(k)$ along $I_R(\mathcal{H}_o^S)$ are 0 and $-\nu(S,k)$ with multiplicities rk$(R)-1$ and 1 respectively.

Proof: The fibres of ι_R on V^{reg} are orbits of a cyclic group. Because the system $\mathcal{E}_v(k)$ is homogeneous of degree $\nu(R,k)$ it is invariant under this cyclic group. Hence the push forward $\mathcal{E}_y^Q(k)$ has the same determination order (rk(R)). The homogeneous degree of $\mathcal{E}_y^Q(k)$ is the quotient of the homogeneous degrees of $\mathcal{E}_v(k)$ and the map ι_R. This proves 1.

Properties 2 and 3 are clear. Let S be as in property 4. Recall that in this case $I_R(\mathcal{H}_o^S)$ has codimension one in $Q(R,k)$ and is isomorphic to $Q(S,k)$. Let $u_1 \in V_S$ and $u_2 \in \mathcal{H}_o^S$. The curve

$$\mu \mapsto \Phi_S(u_1, \mu\gamma(u_2))$$

is a smooth curve for $\mu \in \mathbb{C}$ near 0 and passes through $I_R(\mathcal{H}_o^S)$. By definition of Φ_S it is also given by

$$\mu \mapsto c^{st}\iota_R(\mu^{-\frac{\nu(S,k)}{\nu(R,k)}}(\mu u_1 + u_2))$$

where c^{st} is some constant. Recall that the system $\mathcal{E}_v(k)$ is homogeneous of degree $\nu(R,k)$ and has local exponents $\nu(S,k)$ and 0 along V^S with multiplicities $\mathrm{rk}(R)-1$ and 1 respectively. If β is one of these exponents then the formula above shows that $\beta - \nu(S,k)$ is a local exponent of $\mathcal{E}_y^Q(k)$ along $I_R(\mathcal{H}_o^S)$. $\square$

Let A be the algebra of $W(R)$ invariant elements in the coordinate ring of the affine variety $Q(R,k)$. Take $Q(R,k)/W(R) := \mathrm{Spec}(A)$ and think of this as a weighted homogeneous affine variety. Let A^+ be the ideal of A of all elements with zero constant term. We will call A^+ the *origin* of the variety $Q(R,k)/W(R)$. There is a canonical projection of $Q(R,k)$ onto $Q(R,k)/W(R)$. For $U \subseteq V$ denote the quotient $I_R(U)/W(R)$ by $I_R^W(U)$. If $S \in \mathcal{R}_o(\emptyset, R)$ and $\nu(S,k) = 0$ we call $I_R^W(\mathcal{H}_o^S)$ also a *cuspidal line*.

The map ι_R is not injective on V^{reg} and hence $W(R)$ will not even act freely on $\iota_R(V^{reg})$ in general. However we assumed that the rank of R is at least three. A consequence of this is that if $w \in W(R)$ has a fixed point on $\iota_R(V^{reg})$ then the fixed point set of w has codimension at least two in $\iota_R(V^{reg})$ (recall that we excluded the case D_n (n odd) which would be a counter example to this observation). We denote the maximal subset of $\iota_R(V^{reg})$ on which $W(R)$ acts freely by $Q^f(R,k)$. In particular $\iota_R(V^{reg} \cap E) \subset Q^f(R,k)$.

Let Y denote the $W(R)$-orbit of y on $Q^f(R,k)$. The system $\mathcal{E}_y^Q(k)$ descends naturally to a system $\mathcal{E}_Y^Q(k)$ on $Q^f(R,k)/W(R)$. Denote the dual of $\mathcal{E}_Y^Q(k)$ as a vector space by $\mathcal{E}_Y^Q(k)^*$. Analytic continuation of (compound) germs in $\mathcal{E}_Y^Q(k)$ induces a (left) representation

$$\rho^* : \pi_1(Q^f(R,k)/W(R), Y) \to \mathrm{End}(\mathcal{E}_Y^Q(k)^*).$$

Note that $\pi_1(Q^f(R,k)/W(R), Y)$ is isomorphic to $\pi_1(\mathbb{C}^n \backslash \Delta, P(v))$ and hence to $B(M)$. The ρ^*-invariant Hermitian form H^* on $\mathcal{E}_Y^Q(k)^*$ is non degenerate and has signature $(1, n-1)$.

There is a natural multivalued evaluation map ev of $Q^f(R,k)/W(R)$ into $\mathcal{E}_Y^Q(k)^*$ whose monodromy is given by ρ^*. Recall that ev maps even into $\mathbb{B}$, the set of all vectors v such that $H^*(v,v) > 0$. Let B be the $(n-1)$-dimensional complex ball. Then $\mathbb{C} \times B$ is the universal covering of $\mathbb{B}$. If $\widetilde{X}$ is the universal covering of $Q^f(R,k)/W(R)$ then ev induces a (single valued) map $\widetilde{\mathrm{EV}}$ on $\widetilde{X}$ mapping into $\mathbb{C} \times B$ as in section 3.7. Let $\tilde{\rho} : \mathrm{Aut}(\widetilde{X} \mid X) \to \widetilde{G}$ be a homomorphism onto a group of transformations of $\mathbb{C} \times B$ such that

$$\widetilde{\mathrm{EV}}(gx) = \tilde{\rho}(g)\widetilde{\mathrm{EV}}(x)$$

for all $x \in \widetilde{X}$ and $g \in \mathrm{Aut}(\widetilde{X} \mid X)$ (compare with figure 3.4, page 72).

We can now formulate the main theorem of this section. For $\alpha \in R$ define $p_\alpha := 2/(1-2k_\alpha)$. Let $S \in \mathcal{R}_o(\emptyset, R)$ be of rank $\mathrm{rk}(R)-1$ such that $\nu(S,k) < 0$. Define p_S as $-2/\nu(S,k)$ or $-1/\nu(S,k)$ depending on whether or not $W(R)$ contains an element w stabilizing S such that w or $-w$ is a reflection of V_R.

Theorem 4.15 Assume that the four conditions on page 104 hold. *Suppose that for all $\alpha \in R$ the number $p_\alpha \in \mathbb{N}_{\geq 2}$ and for all S of rank $rk(R)-1$ with negative exponent the number $p_S \in \mathbb{N}_{\geq 1}$. Let $X_u(p)$ be the universal Galois covering of $Q^f(R,k)/W(R)$ with local degrees p_α and p_S along $I_R^W(\alpha^\perp \cap V^{reg})$ and $I_R^W(\mathcal{H}_o^S)$ respectively. Then $\widetilde{EV}$ induces an embedding of $X_u(p)$ into $\mathbb{C}\times B$. Moreover $\mathbb{C}\times B$ is a ramified covering of $Q(R,k)/W(R)$ minus the origin and all cuspidal lines extending $X_u(p)$ and with automorphism group $\widetilde{G}$. In particular the image of ρ^* acts discretely on $\mathbb{B}$.*

Proof: A specialization of the argument from section 3.7. Considering the local exponents of the system $\mathcal{E}_Y^Q(k)$ shows that $\widetilde{\mathrm{EV}}$ descends to a locally biholomorphic map ev_u on $X_u(p)$.

Now use the local structure of $Q(R,k)$ to extend $X_u(p)$ as follows. Let $S \in \mathcal{R}_o(\emptyset, R)$ be such that $\nu(S,k) < 0$. Take x in the relative interior $\iota_S(\mathcal{H}_o^\emptyset(S))$ of $Q(S,k)$ embedded in $Q(R,k)$ by $\phi_{S,R}^*$. By our assumptions on $Q(R,k)$ and the properties of the morphism Φ_S introduced before x has a neighborhood U isomorphic to

$$\Delta^{\mathrm{rk}(S)} \times (\Delta^{N(S,R)} \cap \mathrm{Cone}(S,R)).$$

We may assume that U is such that for any $w \in W(R)$ if $wU \cap U \neq \emptyset$ then w fixes x. We may also assume that $U \cap Q^f(R,k)$ is the product of its projections on each of the factors of U (in the cartesian product for U shown above).

An enumeration of all possible multiplicity parameters k under consideration shows that the pair (S,R) will never be any of those listed in theorem 4.5, page 90. Hence theorem 4.7, page 94 will be applicable to Cone(S,R). See the tables in chapter 5.

Let $S' \subset S$ be the set of roots such that $s_\alpha x = x$ for all $\alpha \in S'$. Then S' is a parabolic root system and each irreducible component has a positive exponent, i.e. is elliptic.

Let the subgroup Σ_x of $W(R)$ be the direct product of $W(S')$ and $W(S,R)$. Here $W(S,R)$ is as introduced in section 4.3. It is a normal subgroup of the stabilizer of x in $W(R)$.

Both $W(S')$ and $W(S,R)$ act on a separate factor in the cartesian product of U. Indeed if $w \in W(S,R)$ such that $w(v) = -v$ for all $v \in V_S$ and $P \in A_N(R,k)$ then

$$P(w(v_S + v^S)) = P(-v_S + w(v^S)) = P(v_S - w(v^S))$$

for all $v_S \in V_S$ and $v^S \in V^S$. This is the reason for introducing the $W(S,R)$-action as we did in definition 4.2, page 86.

Replace U by the smaller symmetric neighborhood

$$U := \cap_{w \in \Sigma_x} wU.$$

The space $(U \cap I_R(V^{reg}))/\Sigma_x$ is also a cartesian product, namely of the complement of a discriminant and the regular part of $\mathrm{Cone}(S,R)/W(S,R)$ intersected with a neighborhood of its origin. Hence the universal Galois covering $U(p)$ of this quotient space with local degrees p_α and p_S along codimension one divisors on $Q(R,k)$ has finite degree and embeds in a smooth ramified covering of U/Σ_x by the results of sections 3.5 and 4.3.

Replace U again by the smaller neighborhood

$$U := \cap_{w \in \mathrm{Stab}_{W(R)}(x)} wU.$$

Then the Galois covering $U(p)$ is also the universal Galois covering with the same local degrees of

$$(U \cap I_R(V^{reg}))/\mathrm{Stab}_{W(R)}(x).$$

Indeed the map of U/Σ_x onto $U/\mathrm{Stab}_{W(R)}(x)$ is a ramified covering with local degrees one along the codimension one divisors.

Again it can be shown (using the fact that $\widetilde{\mathrm{EV}}$ induces a locally biholomorphic map on the extension of $U(p)$ as in theorem 3.14, page 58 in section 3.5) that $U(p)$ embeds in $X_u(p)$ and hence this covering extends to a ramified covering $X_r^*(p)$ of $Q(R,k)/W(R)$ minus the cuspidal lines and the origin. A similar argument as in section 3.7 shows that the map ev_u extends to a biholomorhic map of $X_r^*(p)$ onto $\mathbb{C} \times B$.

This proves the theorem. $\square$

4.6 Some computational results

Let k be a rational multiplicity parameter and $N > 0$ a common denominator of k. In this section we study a certain subalgebra of $\mathcal{A}_N(R,k)$. Let $m : R \to \mathbb{N}$ be some multiplicity parameter (not necessarily $W(R)$-invariant). The following "monomial"

$$\prod_{\alpha>0} (\alpha^*)^{m_\alpha}$$

is an element of $\mathcal{A}_N(R,k)$ if and only if it satisfies:

1. $\sum_{\alpha>0} m_\alpha = -dN\nu(R,k)$ for some $d \in \mathbb{N}$.
2. $\sum_{\alpha \in S \cap R^+} m_\alpha \geq -dN\nu(S,k)$ for all $S \in \mathcal{R}_o(\emptyset, R)$ and $d \in \mathbb{N}$ satisfying property 1.

All such monomials together generate a graded subalgebra $\mathcal{A}_N^r(R,k)$ of $\mathcal{A}_N(R,k)$.

Theorem 4.16 *If R is of type A_n then $\mathcal{A}_N^r(R,k)$ and $\mathcal{A}_N(R,k)$ are the same.*

Proof: This is a consequence of the main theorem in invariant theory for $SL(2,\mathbb{C})$: The algebra of invariant sections in $\mathcal{L}$ is generated by products of determinants $\det(y_i\ y_j)$, $i \neq j$, $\mathbf{y} \in \mathrm{Mat}(2 \times (n+2), \mathbb{C})$. Under the isomorphism $(\gamma_d) : A(\mathcal{L}) \to \mathcal{A}_N(A_n, m/N)$ these determinants coincide with dual roots α^*. $\square$

Because $\mathcal{A}_N^r(R,k)$ can be identified with the $\mathbb{C}$-algebra generated by a rational cone in $\mathbb{N}^{R^+}$ it follows that $\mathcal{A}_N^r(R,k)$ is finitely generated.

Theorem 4.17 *The algebra $\mathcal{A}_N^r(R,k)$ is non trivial exactly in the following cases:*

R	$\nu(R,k) \in$
A_n	$[-1,0)$
B_n	$[-2,0)$
E_6	$[-3,0)$
E_7	$[-7/2,0)$
E_8	$[-16/3,0)$
F_4	$[-4,0)$
H_3	$[-3,0)$
H_4	$[-8,0)$

Proof: Suppose the monomial with multiplicity parameter $m \neq 0$ is an element of $\mathcal{A}_N(R,k)$. Summing up all multiplicity inequalties for parabolic irreducible root subsystems of corank one in R yields lower bounds for $\nu(R,k)$ as reproduced in the table. Of course 0 is an upper bound for $\nu(R,k)$.

Take for example $R = H_3$. There are six root subsystems of type $I_2(5)$ and every root is contained in exactly two of those. This gives

$$-2dN\nu(H_3,k) = 2\sum_{\alpha>0} m_\alpha =$$

$$\sum_{S \text{ of type } I_2(5)} \left(\sum_{\alpha \in S \cap H_3^+} m_\alpha \right) \geq -6dN\nu(I_2(5),k) = -3dN(\nu(H_3,k)+1)$$

and thus a lower bound for $\nu(H_3,k)$ of -3.

In every case satisfying the bounds listed above one can explicitly construct a non constant monomial in $\mathcal{A}_N(R,k)$. For example if m is any multiple of $-N\nu(H_3,k)$ then the monomial

$$\prod_{\alpha \in H_3^+} (\alpha^*)^m$$

is an element of $\mathcal{A}_N^r(H_3,k)$. This proves the theorem. $\square$

It is easy to check that the homomorphisms $\phi_{S,R}$ map $\mathcal{A}_N^r(R,k)$ into $\mathcal{A}_N^r(S,k)$ and hence restrict to homomorphisms $\phi_{S,R}^r$.

Theorem 4.18 *If R is of type A_n or B_n and $\nu(R,k)$ lies in the hyperbolic range then all $\phi^r_{S,R}$ are surjective. In any of the cases (F_4,p,q) with $p=2$, $q \le 12$ or $p=3$ and $q \in \{3,4,6,12\}$ or $p=4$ and $q=4$ the homomorphisms $\phi^r_{S,R}$ are surjective. In the latter cases we take the multiplicity parameter k such that $k_R = \{1/2-1/p, 1/2-1/q\}$.*

Proof: In fact in all these cases a monomial in $\mathcal{A}^r_N(S,k)$ has a monomial preimage in $\mathcal{A}^r_N(R,k)$. It suffices to consider S of corank one in R.

Suppose R is of type A_n and S is of type A_{n-1}. Let $m \neq 0$ be a multiplicity parameter on S such that the corresponding monomial is an element of $A_N(R,k)$ say. If m' is a multiplicity parameter on A_n such that its restriction to S is m then it is not hard to check that the monomial corresponding to m' is an element of $A_N(R,k)$ if and only if for every $\beta \in A_n \backslash S$

$$m'_\beta \le \left(\sum_{\alpha \perp \beta} m_\alpha \right) + N\nu(A_{n-2},k)$$

and of course $\sum_{\alpha \in A_n} m_\alpha = -N\nu(A_n,k)$. Note that roots perpendicular to β form a system of type A_{n-2} contained in S. All these inequalities can indeed be fulfilled exactly if $-1 \le \nu(A_n,k) < 0$.

The case $R = B_n$ can be treated with a similar argument involving root subsystems of corank one and two.

The listed cases for F_4 were checked on a computer. This was done by only considering the extremal multiplicity parameters on corank one susbsystems. The computation then amounts to a feasibility test of a set of linear inequalities. □

If in any of the cases in theorem 4.18 $\nu(R,k)$ is *strictly* greater than the lower bounds listed in the table above then there exists a multiplicity parameter m such that all inqualities on m corresponding to root subsystems are *strict* inequalities. This implies in particular that $\mathcal{A}^r_N(R,k)$ has sufficiently many elements to ensure that ι_R is an immersion on V^{reg} with cyclic orbits as fibres.

We conclude this chapter with a final remark.

Remark 4.7 *Suppose R and k are such that $\nu(S,k) > 0$ for every parabolic irreducible root subsystem S of corank at least two in R. Then instead of considering $Q(R,k)$ it suffices for the purpose of proving the main theorem to blow up V_R in all one dimensional linear subspaces with a* hyperbolic *stabilizer.*

In particular this suffices to handle all cases where R has rank three. Also $(H_4,3)$ and (F_4,p,q) with $p=2$ and $q=4,5$ or $p=3$ and $q=4$ are other examples. See the tables in the next chapter.

It is an interesting question if in general the variety $Q(R,k)$ (or one with similar properties) can be obtained by succesive blow ups and blow downs of V_R.

4.7 Literature

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Chapter 5

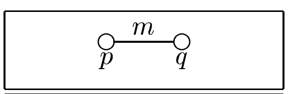
Tables

5.1 The marked Coxeter diagrams

This chapter contains the tables of marked Coxeter diagrams of elliptic, parabolic and hyperbolic type for which the associated complex reflection group is discrete in the suitable unitary group. For some hyperbolic diagrams discreteness is still conjectural (see the remark in section 5.4). The tables list all cases of rank at least two and with a mark that is at least three.

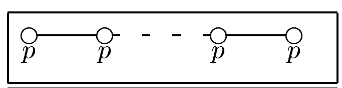
In the elliptic case the associated reflection group is finite. In the parabolic case it acts cocompactly on affine space. For hyperbolic diagrams the associated reflection group acts discretely on the complex hyperbolic ball. In the hyperbolic case the action is cocompact for all diagrams that do not contain parabolic subdiagrams. In all other cases it acts with cofinite volume.

5.2 The elliptic diagrams



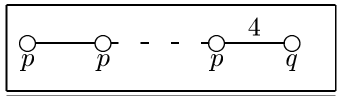
m	5	6	8	10
p	3	2	2	2
q	3	$3, 4, 5$	3	3

Type $(I_2(m), p, q)$



rk	2	3	4
p	$3, 4, 5$	3	3

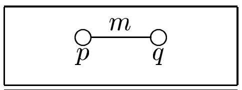
Type (A_{rk}, p)



rk	≥ 2	2	3
p	2	3	3
q	≥ 3	$3, 4, 5$	2

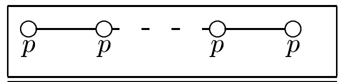
Type (B_{rk}, p, q)

5.3 The parabolic diagrams



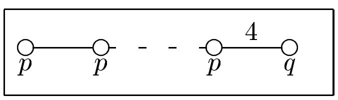
m	6		8	12
p	2	3	2	2
q	6	3	4	3

Type $(I_2(m), p, q)$



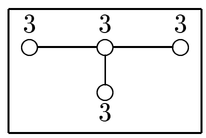
rk	2	3	5
p	6	4	3

Type (A_{rk}, p)

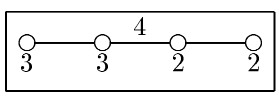


rk	2		3		4
p	3	4	3	4	3
q	6	4	3	2	2

Type (B_{rk}, p, q)



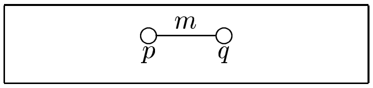
Type $(D_4, 3)$



Type $(F_4, 3, 2)$

5.4 The hyperbolic diagrams

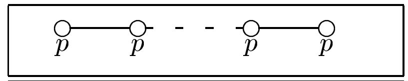
Diagrams that contain a hyperbolic proper subdiagram are the result of the theory in chapter 4.



$\frac{1}{p} + \frac{1}{q} < 1 - \frac{2}{m}$

$m \geq 5$ and $p = q$ if m is odd.

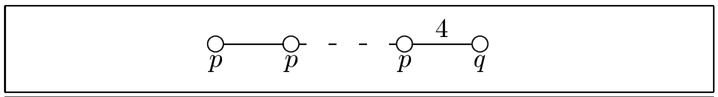
Type $(I_2(m), p, q)$



rk	2	3
p	≥ 7	$5, 6, 7, 8, 9, 10, 12, 18$

rk	4	5	6	7	8	9
p	$4, 5, 6, 8$	$4, 5$	3	3	3	3

Type (A_{rk}, p)



rk	3	
p	3	4
q	$4, 5, 6, 7, 8, 9, 10, 12, 15, 18, 24, 42$	$3, 4, 5, 6, 8, 12, 20$

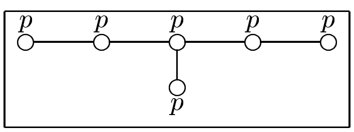
p	5	6	7	8
q	$2, 3, 4, 5, 10, 20$	$2, 3, 4, 6, 12$	$2, 3, 42$	$2, 3, 4, 8, 24$

p	9	10	12	18
q	$2, 3, 18$	$2, 3, 5, 15$	$2, 3, 4, 6, 12$	$2, 3, 9$

rk	4				
p	3	4	5	6	8
q	$3, 4, 6, 12$	$2, 3, 4, 6$	2	$2, 3, 6$	$2, 4$

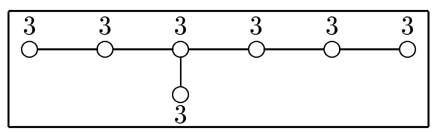
rk	5		6		7	8
p	3	4	3	4	3	3
q	$2, 3, 4, 6$	$2, 4$	$2, 3, 6$	2	$2, 3$	2

Type (B_{rk}, p, q)

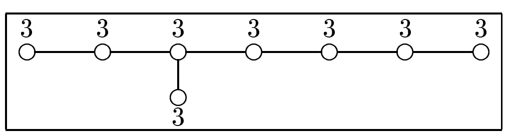


p	$3, 4$

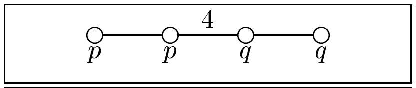
Type (E_6, p)



Type $(E_7, 3)$

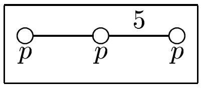


Type $(E_8, 3)$



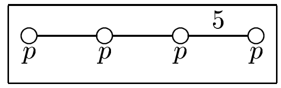
p	2	3	4	6
q	$4, 5, 6, 8, 12$	$3, 4, 6, 12$	4	6

Type (F_4, p, q)



p	$3, 4, 5, 10$

Type (H_3, p)



p	$3, 5$

Type (H_4, p)

Index of notations

Index

Complexe spiegelingsgroepen en hypergeometrische funkties

In de theorie van eindige reële spiegelingsgroepen zijn de resultaten betreffende presentaties en invariantentheorie van dergelijke groepen nadrukkelijk aanwezig.

Voor eindige (eventueel complexe) spiegelingsgroepen in het algemeen is de invariantentheorie evenzeer goed begrepen (in deze theorie is het niet van belang om de ordes van de voortbrengende spiegelingen te kennen). Op het gebied van presentaties van deze algemenere groepen ligt dat anders. Hier zijn presentaties beschreven door deze met een computer geval voor geval te testen, hetgeen in essentie mogelijk is daar de betrokken groepen eindig zijn.

In dit proefschrift wordt van een zekere klasse van complexe spiegelingsgroepen (waaronder zowel eindige (Shephard-groepen) als niet-eindige groepen vallen) op een intrinsieke manier resultaten bewezen betreffende presentaties en invarianten.

Belangrijkste hulpmiddel bij het opzetten van deze theorie zijn de hypergeometrische funkties geassocieerd met wortelsystemen. In het bijzonder de algebraïsche en meetkundige kant van het analytisch voortzettingsgedrag wordt uitgebreid bestudeerd.

Hoofdstuk 1 schetst de gevolgde methoden aan de hand van de symmetrische groep. Hoofdstuk 2 tracht reeds bekend werk van Deligne en Mostow in meer elementaire termen uiteen te zetten. Hoofdstuk 3 vormt in wezen de kern van dit proefschrift en behandelt willekeurige eindige wortelsystemen met daaraan gerelateerde complexe spiegelingsgroepen. Hoofdstuk 4 tenslotte is een aanzet om resultaten van hoofdstuk 3 in een algemenere vorm te kunnen begrijpen en bewijzen. Dit laatste hoofdstuk is voornamelijk meetkundig van aard.

Curriculum Vitae

Ik ben op 4 juli 1968 te Nijmegen geboren waar ik van 1980 to 1986 het atheneum doorliep aan het Dominicus College. Van 1986 tot 1990 studeerde ik wiskunde aan de Katholieke Universiteit Nijmegen en behaalde het doctoraalexamen cum laude. Aansluitend werd ik Assistent In Opleiding bij de vakgroep wiskunde van de KUN in het gebied Algebra en Meetkunde. Onder begeleiding van dr. G.J. Heckman verrichtte ik onderzoek in voornamelijk Lie-theorie en theorie van speciale funkties. Een deel van de resultaten van dit onderzoek is gebundeld in dit proefschrift. Vanaf september 1994 ben ik werkzaam als applicatieprogrammeur bij een Nijmeegs automatiseringsbedrijf.